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Third derivative hybrid block integrator for solution of stiff systems of initial value problems

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Abstract A new third derivative hybrid block method is presented for the solution of first order stiff systems of initial value problems. The main method and additional methods are obtained from the same continuous scheme derived via interpolation and collocation procedures using power series as the basis function. The continuous representation of the scheme permits us to evaluate at both grid and off-grid points. The stability properties of the method is discussed. The block method is applied simultaneously to generate the numerical solutions of (1) over non-overlapping intervals. Numerical results obtained using the proposed third derivative hybrid method in block form reveal that it compares favorably well with existing methods in the literature.

Keywords Block hybrid method · Off-step points · Collocation and interpolation · Stability

Mathematics Subject Classification 65L05 · 65L06

1 Introduction

Consider the initial value problems (IVPs) of the form

$$y' = f(t, y), \ \ y(t_0) = y_0 \tag{1}$$

where $f : \Re \times \Re^{3d} \longrightarrow \Re^d$; y, $y_0 \in \Re^d$, and d is dimension of the system. f satisfies the conditions which guarantee that the problem has a unique continuously differentiable solution, which we denote by y(t) (see Henrici [1]), and the Jacobian $(\frac{\partial x}{\partial y})$ whose eigenvalues have negative real parts varies slowly ([2]). These type of equations arise frequently in engineering, science, and biological sciences. It is common knowledge that a vast number of these problems cannot be solved analytically, hence the need for the numerical methods

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for such problems remains crucial. In the literature, several authors have proposed various techniques for the numerical solution of (1) which include linear multistep methods (LMMs). It should be noted that (1) is efficiently solved by A-stable methods and for high accuracy, higher order methods are preferable. However, for linear multistep methods (LMMs), the use of high order LMMs for (1) is restricted by the second Dahlquist [3] barrier theorem which stated that the order of an A-stable linear multistep method cannot exceed 2. Several methods have been proposed to overcome this barrier theorem, for instance, (see Akinfenwa et al. ([4–9]), Gear [10], Gragg and Stetter [11], Butcher [12], Lambert [13], and Kohfeld and Thompson [14]), the second derivative methods (see Enright [15], Gupta [16], and Hairer and Wanner [17], and exponentially fitted methods (see Jackson and Kenue [2], Cash [18]). Hybrid method (2) is the modified form of the k-step linear multistep method (LMM) obtained by incorporating off-step points in the derivation process in order to overcome the Dahlquist barrier theorem.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_{\eta j} f_{n+\eta j} \tag{2}$$

Gupta [16] noted that the design of algorithms for hybrid methods is more tedious due to the occurrence of off-steps function in the methods which increases the number of predictors needed to implement the methods. Like ([6–9, 19, 20]), the block hybrid method (3) proposed in this paper is developed via collocation and interpolation procedure, it's self-starting and implemented without the use of predictors.

$$y_{n+k} = y_{n+k-1} h \left(\sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{i=1}^{v} \beta_{\eta j} f_{n+\eta j} \right) + h^2 \gamma_k g_{n+k} + h^3 \zeta_k \tau_{n+k} \tag{3}$$

where h is the stepsize, $k = 2$ is the step number, $v = 2$ is the number of off-points, ηi, $i = 1, 2$ are rational number, β_j, $\beta_{\eta j}$, γ_k, and ζ_k are unknown coefficients that must be determined.

The rest of the paper is presented as follows: in Sect. 2, we discuss the basic idea behind the method and obtain a continuous representation $Y(t)$ for the exact solution $y(t)$ which is used to generate members of the block method for solving (1). In Sect. 3, we present the analysis of newly developed two step third derivative hybrid block method. In Sect. 4, we show the accuracy of our method. Finally, in Sect. 5 we present some concluding remarks.

2 Derivation of the method

In order to obtain (3), we proceed by seeking an approximation of the exact solution $y(t)$ by assuming a continuous solution $Y(t)$ of the form

$$Y(t) = \sum_{j=0}^{3k+1} b_j \varphi_j(t) \tag{4}$$

where $t \in (t_0, T_n)$, b_j are unknown coefficients to be determined, $\varphi_j(t)$ are polynomial basis functions of degree $3k + 1$. Since this polynomial must pass through the interpolation points $[t_{n+k-1}, y_{n+k-1}]$ and the collocation points $[t_n, y_n), (t_{n+1}, y_{n+1}), \ldots (t_{n+k}, y_{n+k}]$ we require that the following $(3k + 2)$ equations must be satisfied.

$$\sum_{j=0}^{7} b_j t^j = y_{n+i},\ i = 1 \tag{5}$$

$$\sum_{j=0}^{7} b_j j t_{n+i}^{j-1} = f_{n+i},\ i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{6}$$

$$\sum_{j=0}^{7} b_j j(j-1) t_{n+i}^{j-2} = g_{n+i},\ i = k \tag{7}$$

$$\sum_{j=0}^{7} b_j j(j-1)(j-2) t_{n+i}^{j-3} = \tau_{n+i},\ i = k \tag{8}$$

Equations (5), (6), (7) and (8) lead to a system of eight equations which is solved to obtain the coefficients b_j and are then substituted into (2). After some algebraic simplification the continuous representation of the third derivative hybrid method is obtained in the form

$$Y(t) = y_{n+k-1} + h\left(\sum_{j=0}^{k} \beta_j(t) f_{n+j} + \beta_\eta j(t) f_{n+\eta j}\right) + h^2 \gamma_k(t) g_{n+k} + h^3 \zeta_k(t) \tau_{n+k} \tag{9}$$

where $\beta_j(t),\ j = 0, 1, 2,\ \beta_\eta j(t), \eta j = \frac{i}{2}, i = 1, 3,\ \gamma_k(t)$, and $\zeta_k(t)$, are continuous coefficients k is the step number, and h is the chosen step-length. We assume that $y_{n+j} = Y(t_n + jh)$ is the numerical approximation to the analytical solution $y(t_{n+j})$, $y'_{n+j} = f(t_{n+j}, y_{n+j})$ is an approximation to $y'(t_{n+j})$, $y'_{n+\eta j} = f(t_{n+\eta j}, y_{n+\eta j})$ is an approximation to $y'(t_{n+\eta j})$, $g_{n+k} = \frac{df}{dt}(t_{n+k}, y(t_{n+k}))$, and $\tau_{n+k} = \frac{d^2 f}{dt^2}(t_{n+k}, y(t_{n+k}))$.

The same continuous method (9) is then used to generate the main method by evaluating (9) at $t = (t_n, t_{n+2})$ and additional methods at $t = (t_n, t_{n+\frac{1}{2}}, t_{n+1})$, and $t = (t_{n+\frac{3}{2}})$ to yield

$$\begin{aligned} y_{n+2} = y_{n+1} &+ \frac{h}{1120} f_n - \frac{32h}{2835} f_{n+\frac{1}{2}} + \frac{43h}{210} f_{n+1} + \frac{64h}{105} f_{n+\frac{3}{2}} + \frac{17791h}{90720} f_{n+2} \\ &- \frac{17h^2}{3024} g_{n+2} - \frac{h^3}{1008} \tau_{n+2} \end{aligned} \tag{10}$$

$$\begin{aligned} y_n = y_{n+1} &- \frac{493h}{3360} f_n - \frac{736h}{945} f_{n+\frac{1}{2}} + \frac{9h}{70} f_{n+1} - \frac{64h}{105} f_{n+\frac{3}{2}} + \frac{12293h}{30240} f_{n+2} \\ &- \frac{139h^2}{1008} g_{n+2} + \frac{5h^3}{336} \tau_{n+2} \end{aligned} \tag{11}$$

$$\begin{aligned} y_{n+\frac{1}{2}} = y_{n+1} &+ \frac{47h}{17920} f_n - \frac{4387h}{22680} f_{n+\frac{1}{2}} - \frac{1499h}{3360} f_{n+1} + \frac{269h}{840} f_{n+\frac{3}{2}} - \frac{270113h}{1451520} f_{n+2} \\ &+ \frac{2887h^2}{48384} g_{n+2} - \frac{97h^3}{16128} \tau_{n+2} \end{aligned} \tag{12}$$

$$\begin{aligned} y_{n+\frac{3}{2}} = y_{n+1} &+ \frac{59h}{53760} f_n - \frac{101h}{7560} f_{n+\frac{1}{2}} + \frac{243h}{1120} f_{n+1} + \frac{361h}{840} f_{n+\frac{3}{2}} - \frac{65059h}{483840} f_{n+2} \\ &+ \frac{629h^2}{16128} g_{n+2} - \frac{19h^3}{5376} \tau_{n+2} \end{aligned} \tag{13}$$

The block hybrid method is then implemented by simultaneously applying (10), (11), (12), and (13) to provide the approximate solution $y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}$, for $n = 0, k, 2k, \ldots, N-k$ on the partition $[t_0, t_2, t_4, \ldots, t_{N-2}, t_N]$.

3 Analysis of the two step third derivative hybrid block method

In this section, we discuss the local truncation error and order, consistency, zero-stability, and convergence of the two step third derivative hybrid block method.

3.1 Local truncation error and order

Following Fatunla [21] and Lambert [22] we define the local truncation error associated with (3) to be the linear difference operator.

$$
\begin{aligned}
L[y(t);h] = &\sum_{j=0}^{k} \alpha_j y(t+jh) - h\sum_{j=0}^{k} \beta_j y'(t+jh) \\
&-h\sum_{j=1}^{v} \beta_\eta j y'(t+\eta j) - h^2\gamma_k y''(t+kh) - h^3\zeta_k y'''(t+kh) \qquad (14)
\end{aligned}
$$

Assuming that $y(t)$ is sufficiently differentiable, we can expand the terms in (10) as a Taylor series expansion about the point t to obtain the expression

$$L[y(t);h] = C_0 y(t) + C_1 y'(t) + \cdots + C_s h^s y^{(s)}(t) + \ldots,$$

where the constant coefficients $C_s, s = 0, 1, \ldots$ are given as follows:

$$
\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \\
C_1 &= \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j - \sum_{j=1}^{v} \beta_{\eta j} \\
C_2 &= \frac{1}{2!}\left(\sum_{j=1}^{k} j^2\alpha_j - 2\left(\sum_{j=0}^{k} j\beta_j\right) - \sum_{j=1}^{v} \eta j\beta_{\eta j}\right) - \gamma_k \\
C_3 &= \frac{1}{3!}\left(\sum_{j=1}^{k} j^3\alpha_j - 3\sum_{j=0}^{k} j^2\beta_j - 3\sum_{j=1}^{v} \eta j^2\beta_{\eta j}\right) - k\gamma_k - \zeta_k \\
C_4 &= \frac{1}{4!}\left(\sum_{j=1}^{k} j^4\alpha_j - 4\sum_{j=0}^{k} j^3\beta_j - 4\sum_{j=1}^{v} \eta j^3\beta_{\eta j}\right) - \frac{1}{2!}k^2\gamma_k - k\zeta_k \\
&\vdots \\
C_p &= \frac{1}{p!}\sum_{j=1}^{k} j^p\alpha_j - \frac{1}{(p-1)!}\sum_{j=0}^{k} j^{p-1}\beta_j \\
&\quad - \frac{1}{(p-1)!}\sum_{j=1}^{v} \eta j^{p-1}\beta_{\eta j} - \frac{1}{(p-2)!}k^{p-2}\gamma_k - \frac{1}{(p-3)!}k^{p-3}\zeta_k
\end{aligned}
$$

According to Henrici [1], the third derivative hybrid method (3) has order p if

$$L[y(t); h] = O(h^{p+1}), \quad C_0 = C_1 = \cdots = C_p = 0, \; C_{p+1} \neq 0 \tag{15}$$

Therefore, C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(t_n)$ the principal local truncation error at the point t_n. It was established from our calculations that the block hybrid method has high order $(7, 7, 7, 7)^T$, and a relatively small error constants given as $-\frac{1}{56448}$, $\frac{-197}{43352064}$, $\frac{59}{6286896000}$, $\frac{1}{130977000}$ where T is a transpose

3.2 Zero-stability

The Eqs. (10), (11), (12) and (13) can be represented by a matrix finite difference equation given by

$$A^{(1)}Y_{\omega+1} = A^{(0)}Y_{\omega} + hB^{(1)}F_{\omega+1} + hB^{(0)}F_{\omega}h^2D^{(1)}G_{\omega} + h^3E^{(1)}U_{\omega}F_{\omega} \tag{16}$$

where

$$\begin{aligned} Y_{\omega+1} &= (y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2})^T, \quad Y_{\omega} = (y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n)^T, \\ F_{\omega+1} &= (f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2})^T, \quad F_{\omega} = (f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n)^T, \\ G_{\omega+1} &= (g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{3}{2}}, \ldots, g_{n+2})^T, \quad U_{\omega+1} = (\tau_{n+\frac{1}{2}}, \tau_{n+1}, \tau_{n+\frac{3}{2}}, \tau_{n+2})^T, \end{aligned}$$

for $\omega = 0, \ldots$ and $n = 0, k, \ldots, N-k$.
And the matrices $A^{(1)}$, $A^{(0)}$, $B^{(1)}$, $B^{(0)}$, D^1 and $E^{(1)}$ are 4 by 4 matrices whose entries are given by:

$$A^{(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} -\frac{4387}{22680} & -\frac{1499}{3360} & \frac{269}{840} & -\frac{270113}{1451520} \\ -\frac{736}{945} & \frac{9}{70} & -\frac{64}{105} & \frac{12293}{30240} \\ -\frac{101}{7560} & \frac{243}{1120} & \frac{361}{840} & -\frac{65059}{483840} \\ -\frac{32}{2835} & \frac{43}{210} & \frac{64}{105} & \frac{17791}{90720} \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{97}{17920} \\ 0 & 0 & 0 & -\frac{493}{3360} \\ 0 & 0 & 0 & \frac{59}{53760} \\ 0 & 0 & 0 & \frac{1}{1120} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \frac{2887}{48384} \\ 0 & 0 & 0 & -\frac{139}{1008} \\ 0 & 0 & 0 & \frac{629}{16128} \\ 0 & 0 & 0 & -\frac{17}{3024} \end{pmatrix}$$

$$E^{(1)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{97}{16128} \\ 0 & 0 & 0 & \frac{5}{336} \\ 0 & 0 & 0 & -\frac{19}{5376} \\ 0 & 0 & 0 & -\frac{1}{1008} \end{pmatrix}$$

The zero stability of the methods in (16) are determined as the limit h tends to zero. Thus as $h \to 0$ the method (11) tends to the difference system

$$A^{(1)} Y_{\omega+1} - A^{(0)} Y_{\omega}$$

which is normalized to obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(A^{(1)} R - A^{(0)}) = R^3(1 - R) \tag{17}$$

Following Fatunla [21], the block method (16) is zero-stable, since from (14), $\rho(r) = 0$ satisfies $|R_j| \leq= 1$, $j = 1, \ldots, 5$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1. We note that the single members of the block method are not zero-stable, but this property is gained when the methods are combined as numerical integrators in the block form (16).

3.3 Consistency and convergence

The block method (16) is consistent since each of the integrators has order $m > 1$. According to Henrici [1], convergence = consistency + zero-stability. Hence the two step third derivative hybrid block method is convergent.

3.4 Stability analysis

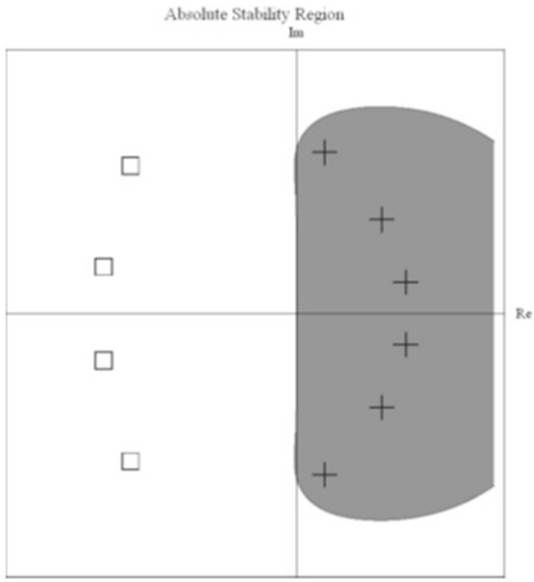
The linear stability properties of the newly derived methods are determined by expressing them in the form (16) and applying them to the test problems $y' = \lambda y$, $y'' = \lambda^2 y$, $y''' = \lambda^3 y$, $\lambda < 0$ to yield

$$Y_{\omega+1} = M(z) Y_{\omega}, \quad z = \lambda h, \tag{18}$$

where the amplification matrix $M(z)$ is given by

$$M(z) = (A^{(1)} - zB^{(1)} - z^2 D^1 - z^3 E^1)^{-1} (A^{(0)} + zB^0) \tag{19}$$

Fig. 1 Stability region



Definition A method is said to be A-stable if (i) all $z \in \mathbb{C}^-$, $M(z)$ has a dominant eigenvalue ϖ_{max} such that $|\varpi_{max}| \leq 1$. More so, since ϖ_{max} is a rational function, the real part of the zeros of ϖ_{max} must be negative, while real part of the poles of ϖ_{max} must be positive; (ii)A_0-stable if for all $z \in \Re \subset \mathbb{C}^-$ $M(z)$ has a dominant eigenvalue ϖ_{max} such that $|\varpi_{max}| \leq 1$; (iii) L_0-stable if it is A_0-stable and $\lim_{z\to-\infty} \varpi_{max} = 0$; (iv) L-stable if it is A-stable and $\lim_{z\to-\infty} \varpi_{max} = 0$

The matrix $M(z)$ has eigenvalues $\{\varpi_1, \varpi_2\varpi_3$ and $\{\varpi_4\} = \{0, 0, 0, \varpi_4\}$ where the dominant eigenvalue ϖ_4 is the stability function $R(z) : \mathbb{C} \to \mathbb{C}$ which is a rational function with real coefficients given by

$$\varpi_4(z) = \frac{3(1680 + 1200z + 350z^2 + 50z^3 + 3z^4)}{5040 - 6480z + 3930z^2 - 1470z^3 + 369z^4 - 62z^5 + 6z^6}$$

with the zeros of $\varpi_4(z)$ denoted with squares having negative real part and the poles denoted with plus signs having positive real part.

Remark Clearly, from Fig. 1, it is obvious that the new method is not A-stable but L_0-stable since from the above definition the method is A_0-stable and $\lim_{z\to-\infty} \varpi_{max} = 0$.

4 Implementation

Method (16) is in block form and is applied in a block-by-block fashion. This is enhanced by the availability of the continuous representation (7), which generates a main discrete hybrid method and three additional methods, which are combined and used as a block method to simultaneously produce approximations $[y_{\frac{1}{2}}, y_1, y_{\frac{3}{2}}, y_2]$ for the solution of (1) at points $[t_{\frac{1}{2}}, t_1, t_{\frac{3}{2}}, t_2]$ in the first block. The new method is implemented more efficiently by combining the hybrid methods as simultaneous integrators for IVPs without requiring starting values and predictors. We proceed by explicitly obtaining initial conditions at

Table 1 Computed values of $error = |y(t) - y|$, at t = 10, using TDHBM for Example 5.1

h	$\varsigma = -10$		$\varsigma = -1000$	
	$Error y_1$	$Error y_2$	$Error y_1$	$Error y_2$
0.1	4.280×10^{-14}	2.973×10^{-14}	1.196×10^{-13}	1.196×10^{-13}
0.05	3.802×10^{-16}	1.966×10^{-16}	1.005×10^{-15}	1.005×10^{-15}
0.025	3.196×10^{-18}	1.304×10^{-18}	8.170×10^{-18}	8.171×10^{-18}
0.0125	2.596×10^{-20}	9.039×10^{-21}	6.514×10^{-20}	6.515×10^{-20}

$t_{n+2}, n = 0, 2, \dots, N-2$ using the computed values $Y(t_{n+2} = y_{n+2})$ over sub-intervals $([t_0, t_2], \dots, [t_{N-2}, t_N])$.

We summarize the process as follows: On the partition
$\pi_N : a = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N = b$
$h = t_{n+1} - t_n, n = 0, 1, \dots, N-1$

Step 1: Choose N, for $k = 2$, $h = (b-a)/N$, the number of blocks $\Gamma = N/2$. Using (11), $n = 0, \varpi = 0$, the values of $[y_{\frac{1}{2}}, y_1, y_{\frac{3}{2}}, y_2]^T$ are simultaneously obtained over the sub-interval $[t_0, t_2]$ as y_0 is known from the IVPs (1).

Step 2: For $n = 2, \varpi = 1$, the values of $[y_{\frac{5}{2}}, y_3, y_{\frac{7}{2}}, y_4]^T$ are simultaneously obtained over the sub-interval $[t_2, t_4]$, as y_2 is known form the previous block.

Step 3: The process is continued for $n = 4, \dots, N-2$ and $\varpi = 2, \dots, \Gamma$ to obtain approximate solutions to (1) on sub-intervals $[t_4, t_6], \dots, [t_{N-2}, t_N]$.

Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional fashion. We note that for linear problems, we solve the IVPs directly from the start with Gaussian elimination using partial pivoting and for nonlinear problems, we use a modified NewtonRaphson method (Table 1).

5 Numerical examples

Example 5.1 Our first example is the non-linear problem.

$$y_1' = -2y_1 + y_2 + 2Sin(t), \quad y_1 = 2$$
$$y_2' = -(\varsigma + 2)y_1 + (\varsigma + 1)(y_2 + Sin(t) - Cos(t)), \quad y_2 = 3$$

With general solution of the system given by

$$y_1(t) = \chi_1 exp(-t) + \chi_2 exp(\varsigma t) + Sin(t)$$

$$y_2(t) = \chi_1 exp(-t) + \chi_2(\varsigma + 2) exp(\varsigma t) + Cos(t)$$

where χ_1 and χ_2 are arbitrary constants. This system has also been solved in ([22,23]), with $\varsigma = -3$ and -1000 in [22] using classical two step BDF method of order 2 and $\varsigma = -10$ and -1000 in [23] using exponentially fitted BDF method of order 3 with the aim of illustrating the phenomenon of stiffness and the numerical consequences of it.The reason is that, if the initial conditions are $y_1(0) = 2$ and $y_2(0) = 3$, the constants χ_1 and χ_2 get the values $\chi_1 = 2$ and $\chi_2 = 0$ and therefore the exact solution becomes independent of ς. The interval [0, 10]

Table 2 Computed values of $Maxerror = Max|y(t) - y|$, using TDHBM for Example 5.1

h	$\varsigma = -10$		$\varsigma = -1000$	
	MaxError	ROC	MaxError	ROC
0.1	1.281×10^{-12}	–	1.307×10^{-12}	–
0.05	9.604×10^{-15}	7.05	9.821×10^{-15}	7.05
0.025	7.358×10^{-17}	7.02	7.521×10^{-17}	7.02
0.0125	5.690×10^{-19}	7.15	5.817×10^{-19}	7.15

Table 3 Absolute errors = $|y_i(T) - y_i|$ at end point $T = 10$ for Example 5.2

T	$BBDF_8 p = 8$		$TDBHM\ p = 7$	
	$Erry_1$	$Erry_2$	$Erry_1$	$Erry_2$
10	4.18×10^{-13}	2.09×10^{-13}	1.53×10^{-15}	7.64×10^{-16}

was considered and the stepsizes used were $h = \frac{1}{10}, \frac{1}{20}$ and $\frac{1}{40}$. In Table 2 we give the absolute errors from the methods at t = 10, for $\varsigma = -10$ (a nonstiff case) and $\varsigma = -1000$ (a stiff case). Comparison will not be fair as both the classical method and exponential method are of order 2 and 3 while the new hybrid method is of order 7. The results show that the method performs well and from Table 2 the rate of convergence is consistent with the order of the method, irrespective of whether the system is stiff or not.

Example 5.2 Next, we consider a well known classical system see [5,24] in the range $0 \leq t \leq 10$

$$y_1' = 998y_1 + 1998y_2, \quad y_1(0) = 1$$
$$y_2' = -999y_1 - 1999y_2, \quad y_2(0) = 1$$

Its exact solution is given by the sum of two decaying exponentials components

$$y_1 = 4e^{-t} - 3e^{-1000t}$$
$$y_2 = -2e^{-t} + 3e^{-1000t},$$

The stiffness ratio is 1 : 1000. In Table 3, we present result for $BBDF_8$ in Akinfenwa et al. [5] and the new $TDBHM$ at the end point $T = 10$ using the step length $h = 0.1$. The TDBHM of order seven performs better than $BBDF_8$ of order eight

Example 5.3 Next, we consider the non-linear system see ([25,26]) in the range $0 \leq t \leq 10$

$$y_1' = \lambda y_1 + y_2^2, \quad y_1(0) = \frac{-1}{\lambda + 2}$$
$$y_2' = -y_2, \quad y_2(0) = 1$$

where $\lambda = 10{,}000$. The exact solution is:

$$y_1 = \frac{-e^{-2t}}{\lambda + 2},$$
$$y_2 = e^{-t}$$

Table 4 Comparison of results for Example 5.3

	t	SDMM [26] at $h = 0.0001$	TDBHM at $h = 0.01$	TDBHM at $h = 0.1$
$Error(y_1)$	3	2.478147×10^{-11}	3.146086×10^{-25}	1.337807×10^{-17}
$Error(y_2)$	3	2.471093×10^{-6}	1.916685×10^{-20}	2.981299×10^{-13}
$Error(y_1)$	5	3.450271×10^{-14}	1.394823×10^{-24}	2.945373×10^{-18}
$Error(y_2)$	5	2.304573×10^{-8}	3.126055×10^{-20}	1.788161×10^{-13}
$Error(y_1)$	10	3.456372×10^{-18}	4.239569×10^{-27}	3.823273×10^{-20}
$Error(y_2)$	10	3.150734×10^{-10}	3.146622×10^{-21}	2.837687×10^{-14}

Table 5 Absolute errors = $|y_i(T) - y_i|$ at end points $T = 5, 40, 70$ at = 0.01 for Example 5.4

T	Method 3.2 in [27] $p = 8$		Method 3.4 in [27] $p = 11$		TDBHM $p = 7$	
	$Erry_1$	$Erry_2$	$Erry_1$	$Erry_2$	$Erry_1$	$Erry_2$
5	4.8198×10^{-5}	1.0083×10^{-1}	2.3725×10^{-7}	8.8134×10^{-1}	4.8511×10^{-17}	6.4681×10^{-9}
40	8.1806×10^{-9}	1.0908×10^{-1}	2.2033×10^{-9}	2.9378×10^{-1}	5.8782×10^{-18}	7.8377×10^{-10}
70	8.7510×10^{-9}	1.1668×10^{-1}	8.593×10^{-10}	1.1456×10^{-1}	4.0277×10^{-18}	5.3703×10^{-10}
100	9.361×10^{-9}	1.2482×10^{-1}	3.351×10^{-10}	4.4677×10^{-2}	5.3649×10^{-19}	7.1532×10^{-11}

We compare the results in [26] which uses second derivation method in predictor corrector mode at h = 0.0001 with the results obtained using the new TDBHM at h = 0.1 and 0.01 at different values of t. The table shows that the new TDBHM obtained superior results than those of [26] even with larger step size h.

Example 5.4 The next example is a highly stiff system see ([27]).

$$y_1' = -10^7 y_1 + 0.075 y_2, \quad y_1(0) = 1$$
$$y_2' = 7500 y_1 - 0.075 y_2, \quad y_2(0) = -1$$

The eigenvalues of the Jacobian of the system are approximately $\lambda_1 = -1.000000000562500 \times 10^6$ and $\lambda_2 = -0.0743749995813$. This problem has been solved [27]. The result of the new method of order seven is compared with that of Yakubu and Markus [27] using second derivative method of order eight and eleven as displayed in the Tables 4 and 5 below.

Example 5.5 Next is the chemistry problem which has been solved by Gear [28], Cash [29], and Yakubu [27],

$$y_1' = -0.013 y_1 - 1000 y_1 y_3, \quad y_1(0) = 1$$
$$y_2' = -2500 y_2 y_3, \quad y_2(0) = 1$$
$$y_1' = -0.013 y_1 - 1000 y_1 y_3 - 2500 y_2 y_3, \quad y_3(0) = 0$$

This problem was solved in the interval $0 \leq t \leq 50$ using the new TDBHM and the result is presented in the Fig. 2 with the numerical value $h = 0.001$ at the end point $T = 10, 20, 30, 40, 50$ presented in the Table 6 below.

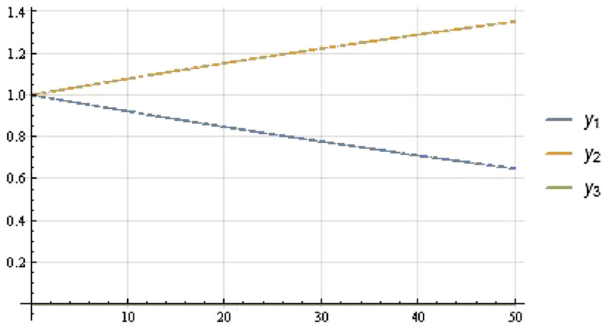


Fig. 2 Graphical solution to Example 5.5

Table 6 Result at end points $T = 10, 20, 30, 40, 50$ for Example 5.5

T	TDBHM		
	y_1	y_2	y_3
10	0.9091683236263698	1.090828425973842	$-3.2503998003423745 \times 10^{-6}$
20	0.8229824068833479	1.177014751859257	$-2.841257419894874 \times 10^{-6}$
30	0.7421209848178718	1.2578765330436872	$-2.4821384719855023 \times 10^{-6}$
40	0.6669652093244602	1.3330326227856673	$-2.167889909722385 \times 10^{-6}$
50	0.5976546980645232	1.4023434085489979	$-1.8933865404310407 \times 10^{-6}$

Table 7 A comparison of methods for number of correct digits Δ, $T = 100$, and $\phi = 10$ for Example 5.6

h	$M(7, r7)\ p = 7$	$TDBHM\ p = 7$
4/5	2.90	6.7
2/5	4.74	9.04
1/5	6.96	10.34
1/10	8.13	13.01
1/20	9.77	13.81
1/40	10.83	13.57

Example 5.6 Finally is the problem whose Jacobian matrix J has purely imaginary eigenvalues on the range $0 \leq t \leq T$

$$y_1' = -\phi y_2 + (1+\phi)Cos(t), \quad y_1(0) = 0$$
$$y_2' = \alpha y_2 - (1+\eta)Sin(t), \quad y_2(0) = 1$$

With exact solution of the system given by $y_1 = sin(t)$, $y_2 = cos(t)$ For any value of the parameter η. Thus, the jacobian J has the following expression

$$J = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$$

the eigenvalues $-i\phi,\ i\phi$.

We compare our method with that of [30] for the correct digit $\Delta = -\log_{10}\left(\frac{||y_i(T) - y_{n,i}||_\infty}{||y_{n,i}||_\infty}\right)$ at the end of the interval for various values of h as shown in Table 7.

6 Conclusion

A two step Third derivative hybrid method is proposed and used together with additional methods in the block form (13) to simultaneously solve (1). The methods are implemented without the need for starting values or predictors and hence complicated subroutines are avoided. The efficiency of the methods have been demonstrated on both linear and non-linear stiff systems of initial value problems. Details of the numerical results are displayed in Tables 1, 2, 3, 4 and 5.

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