

APPENDIX IV: MATHEMATICA SOLUTION FOR SOLVING THE SET OF 12 SIMULTANEOUS NON-LINEAR EQUATIONS

Mathematica has several techniques for solving systems of equations. The most obvious of these methods is Solve, which yields a symbolic exact solution, and NSolve its numeric counterpart. Other non-obvious methods exist including: FindRoot, Minimize, Maximize, NMinimize, NMaximize, and a host of others. To make use of the non-obvious techniques, the problem needs to be recast appropriately. In the solution of this system of equations, the NMinimize and FindMinimum functions were explored. It was discovered the NSolve and Solve break down easily when faced with a non-linear problem. The NMinimize and FindMinimum functions are used to minimize numerically on a set of given parameters. They differ in that FindMinimum attempts to find a local minimum whereas NMinimize attempts to find a global minimum

In[51]:= ?? NMinimize

NMinimize [f , x] minimizes f numerically with respect to x.
 NMinimize [f , { x , y , ... }] minimizes f numerically with respect to x , y , ...
 NMinimize[{f , cons} , {x , y , ...}] minimizes f numerically subject to the constraints cons. >>

Attributes[NMinimize] = {Protected, ReadProtected}

Options [NMinimize] =
 {AccuracyGoal → Automatic, EvaluationMonitor → None, MaxIterations → 100, Method → Automatic, PrecisionGoal → Automatic, StepMonitor → None, WorkingPrecision → MachinePrecision}

In[52]:= ?? FindMinimum

FindMinimum [f , x] searches for a local minimum in f , starting from an automatically selected point.
 FindMinimum [f , { x , x₀ }] searches for a local minimum in f , starting from the point x = x₀.
 FindMinimum [f , { { x , x₀ } , { y , y₀ } , ... }] searches for a local minimum in a function of several variables.
 FindMinimum[{ f , cons } , { { x , x₀ } , { y , y₀ } , ... }] searches for a local minimum subject to the constraints cons.>>
 FindMinimum[{ f , cons } , { x , y , ... }] starts from a point within the region defined by the constraints.

Attributes[FindMinimum] = {HoldAll, Protected}

Options [FindMinimum] = {AccuracyGoal → Automatic, Compiled → Automatic,
 EvaluationMonitor → None, Gradient → Automatic, MaxIterations → Automatic, Method → Automatic, PrecisionGoal → Automatic, StepMonitor → None,
 WorkingPrecision → MachinePrecision}

To make use of these methods appropriately, the problem has to be recast as an optimization problem. The original system of equations has to be transformed into a single function whose zero value will then be sought.

Consider a system of equations as follows:

$$\begin{aligned} F_1(x_1, x_2, \dots, x_n) &= a_1 \\ F_2(x_1, x_2, \dots, x_n) &= a_2 \\ &\vdots \\ F_n(x_1, x_2, \dots, x_n) &= a_n \end{aligned}$$

Where x_1, x_2, \dots, x_n are independent variables and a_1, a_2, \dots, a_n are numeric constants.
 This system of equations can be re-written as:

$$\begin{aligned} F_1(x_1, x_2, \dots, x_n) - a_1 &= 0 \\ F_2(x_1, x_2, \dots, x_n) - a_2 &= 0 \\ &\vdots \\ F_n(x_1, x_2, \dots, x_n) - a_n &= 0 \end{aligned}$$

Or by equating $F_n(x_1, x_2, \dots, x_n) - a_n = G_n(x_1, x_2, \dots, x_n)$

$$\begin{aligned} G_1(x_1, x_2, \dots, x_n) &= 0 \\ G_2(x_1, x_2, \dots, x_n) &= 0 \\ &\vdots \end{aligned}$$

$$G_n(x_1, x_2, \dots, x_n) = 0$$

From the above it can be seen that the solution to the above system of equations is the values of $\{x_1, x_2, \dots, x_n\}$ that minimize the following expression:

$$(G_1^2 + G_2^2 + \dots + G_n^2)$$

The individual functions have been squared to rid any possible negative values. If a solution to the above set of equations exists, then the global minimum value of the above expression should be indeed zero.

This was the method employed in solving the system of equations given below using NMinimize and FindMinimum. While a value of zero was not encountered, the values obtained were close enough to zero. The resulting values for the independent variables were re-substituted into the original set of equations to make comparisons.

In[60]:= eqns = {

$$\frac{k + \frac{g(w_0 + 0.3819w_3)}{|w_0 + 0.3819w_3| + 1} + \frac{e(w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13})}{|w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13}| + 1} + \frac{f(w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23})}{|w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23}| + 1}}{k + \frac{g(w_0 + 0.3819w_3)}{|w_0 + 0.3819w_3| + 1} + \frac{e(w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13})}{|w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13}| + 1} + \frac{f(w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23})}{|w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23}| + 1}} + 1} = 0.3927;$$

$$\frac{k + \frac{g(w_0 + 0.3927w_3)}{|w_0 + 0.3927w_3| + 1} + \frac{e(w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13})}{|w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13}| + 1} + \frac{f(w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23})}{|w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23}| + 1}}{k + \frac{g(w_0 + 0.3927w_3)}{|w_0 + 0.3927w_3| + 1} + \frac{e(w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13})}{|w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13}| + 1} + \frac{f(w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23})}{|w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23}| + 1}} + 1} = 0.3956;$$

$$\frac{k + \frac{g(w_0 + 0.3956w_3)}{|w_0 + 0.3956w_3| + 1} + \frac{e(w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13})}{|w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13}| + 1} + \frac{f(w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23})}{|w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23}| + 1}}{k + \frac{g(w_0 + 0.3956w_3)}{|w_0 + 0.3956w_3| + 1} + \frac{e(w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13})}{|w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13}| + 1} + \frac{f(w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23})}{|w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23}| + 1}} + 1} = 0.3963;$$

$$\frac{k + \frac{g(w_0 + 0.3963w_3)}{|w_0 + 0.3963w_3| + 1} + \frac{e(w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13})}{|w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13}| + 1} + \frac{f(w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23})}{|w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23}| + 1}}{k + \frac{g(w_0 + 0.3963w_3)}{|w_0 + 0.3963w_3| + 1} + \frac{e(w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13})}{|w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13}| + 1} + \frac{f(w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23})}{|w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23}| + 1}} + 1} = 0.3972;$$

$$\frac{k + \frac{g(w_0 + 0.3972w_3)}{|w_0 + 0.3972w_3| + 1} + \frac{e(w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13})}{|w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13}| + 1} + \frac{f(w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23})}{|w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23}| + 1}}{k + \frac{g(w_0 + 0.3972w_3)}{|w_0 + 0.3972w_3| + 1} + \frac{e(w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13})}{|w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13}| + 1} + \frac{f(w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23})}{|w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23}| + 1}} + 1} = 0.4011;$$

$$\frac{k + \frac{g(w_0 + 0.4011w_3)}{|w_0 + 0.4011w_3| + 1} + \frac{e(w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13})}{|w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13}| + 1} + \frac{f(w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23})}{|w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23}| + 1}}{k + \frac{g(w_0 + 0.4011w_3)}{|w_0 + 0.4011w_3| + 1} + \frac{e(w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13})}{|w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13}| + 1} + \frac{f(w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23})}{|w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23}| + 1}} + 1} = 0.4116;$$

$$\frac{k + \frac{g(w_0 + 0.4116w_3)}{|w_0 + 0.4116w_3| + 1} + \frac{e(w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13})}{|w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13}| + 1} + \frac{f(w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23})}{|w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23}| + 1}}{k + \frac{g(w_0 + 0.4116w_3)}{|w_0 + 0.4116w_3| + 1} + \frac{e(w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13})}{|w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13}| + 1} + \frac{f(w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23})}{|w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23}| + 1}} + 1} = 0.4208;$$

$$\frac{k + \frac{g(w_0 + 0.4208w_3)}{|w_0 + 0.4208w_3| + 1} + \frac{e(w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13})}{|w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13}| + 1} + \frac{f(w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23})}{|w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23}| + 1}}{k + \frac{g(w_0 + 0.4208w_3)}{|w_0 + 0.4208w_3| + 1} + \frac{e(w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13})}{|w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13}| + 1} + \frac{f(w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23})}{|w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23}| + 1}} + 1} = 0.4225;$$

$$\frac{k + \frac{g(w_0 + 0.4225w_3)}{|w_0 + 0.4225w_3| + 1} + \frac{e(w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13})}{|w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13}| + 1} + \frac{f(w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23})}{|w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23}| + 1}}{k + \frac{g(w_0 + 0.4225w_3)}{|w_0 + 0.4225w_3| + 1} + \frac{e(w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13})}{|w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13}| + 1} + \frac{f(w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23})}{|w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23}| + 1}} + 1} = 0.4423;$$

$$\frac{k + \frac{g(w_0 + 0.4423w_3)}{|w_0 + 0.4423w_3| + 1} + \frac{e(w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13})}{|w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13}| + 1} + \frac{f(w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23})}{|w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23}| + 1}}{k + \frac{g(w_0 + 0.4423w_3)}{|w_0 + 0.4423w_3| + 1} + \frac{e(w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13})}{|w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13}| + 1} + \frac{f(w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23})}{|w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23}| + 1}} + 1} = 0.4523;$$

$$\frac{k + \frac{g(w_0 + 0.4523w_3)}{|w_0 + 0.4523w_3| + 1} + \frac{e(w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13})}{|w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13}| + 1} + \frac{f(w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23})}{|w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23}| + 1}}{k + \frac{g(w_0 + 0.4523w_3)}{|w_0 + 0.4523w_3| + 1} + \frac{e(w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13})}{|w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13}| + 1} + \frac{f(w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23})}{|w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23}| + 1}} + 1} = 0.4683;$$

$$\frac{k + \frac{g(w_0 + 0.4683w_3)}{|w_0 + 0.4683w_3| + 1} + \frac{e(w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13})}{|w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13}| + 1} + \frac{f(w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23})}{|w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23}| + 1}}{k + \frac{g(w_0 + 0.4683w_3)}{|w_0 + 0.4683w_3| + 1} + \frac{e(w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13})}{|w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13}| + 1} + \frac{f(w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23})}{|w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23}| + 1}} + 1} = 0.6289;$$

vars = {e, f, g, k, w₀, w₂₁, w₂₂, w₂₃, w₃, w₁₁, w₁₂, w₁₃};

The above code initializes the set of equations and declares the independent variables. The following code transposes the equations so they are of the form as G_n above.

In[62]:=

transeqns = Subtract @@ eqns

Out[62]=

{

$$\begin{aligned}
& \left(\frac{k + \frac{g(w_0 + 0.3819w_3)}{|w_0 + 0.3819w_3| + 1} + \frac{e(w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13})}{|w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13}| + 1} + \frac{f(w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23})}{|w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23}| + 1}}{k + \frac{g(w_0 + 0.3819w_3)}{|w_0 + 0.3819w_3| + 1} + \frac{e(w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13})}{|w_0 + 0.3205w_{11} + 0.4591w_{12} + 0.3805w_{13}| + 1} + \frac{f(w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23})}{|w_0 + 0.3793w_{21} + 0.5774w_{22} + 0.3372w_{23}| + 1}} - 0.3927 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.3927w_3)}{|w_0 + 0.3927w_3| + 1} + \frac{e(w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13})}{|w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13}| + 1} + \frac{f(w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23})}{|w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23}| + 1}}{k + \frac{g(w_0 + 0.3927w_3)}{|w_0 + 0.3927w_3| + 1} + \frac{e(w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13})}{|w_0 + 0.3274w_{11} + 0.378w_{12} + 0.4512w_{13}| + 1} + \frac{f(w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23})}{|w_0 + 0.3205w_{21} + 0.4591w_{22} + 0.3805w_{23}| + 1}} - 0.3956 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.3956w_3)}{|w_0 + 0.3956w_3| + 1} + \frac{e(w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13})}{|w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13}| + 1} + \frac{f(w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23})}{|w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23}| + 1}}{k + \frac{g(w_0 + 0.3956w_3)}{|w_0 + 0.3956w_3| + 1} + \frac{e(w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13})}{|w_0 + 0.3454w_{11} + 0.5032w_{12} + 0.4744w_{13}| + 1} + \frac{f(w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23})}{|w_0 + 0.3274w_{21} + 0.378w_{22} + 0.4512w_{23}| + 1}} - 0.3963 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.3963w_3)}{|w_0 + 0.3963w_3| + 1} + \frac{e(w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13})}{|w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13}| + 1} + \frac{f(w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23})}{|w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23}| + 1}}{k + \frac{g(w_0 + 0.3963w_3)}{|w_0 + 0.3963w_3| + 1} + \frac{e(w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13})}{|w_0 + 0.4349w_{11} + 0.4935w_{12} + 0.5626w_{13}| + 1} + \frac{f(w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23})}{|w_0 + 0.3454w_{21} + 0.5032w_{22} + 0.4744w_{23}| + 1}} - 0.3972 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.3972w_3)}{|w_0 + 0.3972w_3| + 1} + \frac{e(w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13})}{|w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13}| + 1} + \frac{f(w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23})}{|w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23}| + 1}}{k + \frac{g(w_0 + 0.3972w_3)}{|w_0 + 0.3972w_3| + 1} + \frac{e(w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13})}{|w_0 + 0.4209w_{11} + 0.5155w_{12} + 0.5531w_{13}| + 1} + \frac{f(w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23})}{|w_0 + 0.4349w_{21} + 0.4935w_{22} + 0.5626w_{23}| + 1}} - 0.4011 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4011w_3)}{|w_0 + 0.4011w_3| + 1} + \frac{e(w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13})}{|w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13}| + 1} + \frac{f(w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23})}{|w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23}| + 1}}{k + \frac{g(w_0 + 0.4011w_3)}{|w_0 + 0.4011w_3| + 1} + \frac{e(w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13})}{|w_0 + 0.4028w_{11} + 0.5092w_{12} + 0.5006w_{13}| + 1} + \frac{f(w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23})}{|w_0 + 0.4209w_{21} + 0.5155w_{22} + 0.5531w_{23}| + 1}} - 0.4116 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4116w_3)}{|w_0 + 0.4116w_3| + 1} + \frac{e(w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13})}{|w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13}| + 1} + \frac{f(w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23})}{|w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23}| + 1}}{k + \frac{g(w_0 + 0.4116w_3)}{|w_0 + 0.4116w_3| + 1} + \frac{e(w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13})}{|w_0 + 0.3748w_{11} + 0.4597w_{12} + 0.5014w_{13}| + 1} + \frac{f(w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23})}{|w_0 + 0.4028w_{21} + 0.5092w_{22} + 0.5006w_{23}| + 1}} - 0.4208 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4208w_3)}{|w_0 + 0.4208w_3| + 1} + \frac{e(w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13})}{|w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13}| + 1} + \frac{f(w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23})}{|w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23}| + 1}}{k + \frac{g(w_0 + 0.4208w_3)}{|w_0 + 0.4208w_3| + 1} + \frac{e(w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13})}{|w_0 + 0.367w_{11} + 0.5394w_{12} + 0.4713w_{13}| + 1} + \frac{f(w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23})}{|w_0 + 0.3748w_{21} + 0.4597w_{22} + 0.5014w_{23}| + 1}} - 0.4225 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4225w_3)}{|w_0 + 0.4225w_3| + 1} + \frac{e(w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13})}{|w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13}| + 1} + \frac{f(w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23})}{|w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23}| + 1}}{k + \frac{g(w_0 + 0.4225w_3)}{|w_0 + 0.4225w_3| + 1} + \frac{e(w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13})}{|w_0 + 0.3603w_{11} + 0.3944w_{12} + 0.4508w_{13}| + 1} + \frac{f(w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23})}{|w_0 + 0.367w_{21} + 0.5394w_{22} + 0.4713w_{23}| + 1}} - 0.4423 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4423w_3)}{|w_0 + 0.4423w_3| + 1} + \frac{e(w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13})}{|w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13}| + 1} + \frac{f(w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23})}{|w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23}| + 1}}{k + \frac{g(w_0 + 0.4423w_3)}{|w_0 + 0.4423w_3| + 1} + \frac{e(w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13})}{|w_0 + 0.324w_{11} + 0.5647w_{12} + 0.4177w_{13}| + 1} + \frac{f(w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23})}{|w_0 + 0.3603w_{21} + 0.3944w_{22} + 0.4508w_{23}| + 1}} - 0.4523 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4523w_3)}{|w_0 + 0.4523w_3| + 1} + \frac{e(w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13})}{|w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13}| + 1} + \frac{f(w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23})}{|w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23}| + 1}}{k + \frac{g(w_0 + 0.4523w_3)}{|w_0 + 0.4523w_3| + 1} + \frac{e(w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13})}{|w_0 + 0.3755w_{11} + 0.4745w_{12} + 0.4911w_{13}| + 1} + \frac{f(w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23})}{|w_0 + 0.324w_{21} + 0.5647w_{22} + 0.4177w_{23}| + 1}} - 0.4683 \right)^2 + \\
& \left(\frac{k + \frac{g(w_0 + 0.4683w_3)}{|w_0 + 0.4683w_3| + 1} + \frac{e(w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13})}{|w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13}| + 1} + \frac{f(w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23})}{|w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23}| + 1}}{k + \frac{g(w_0 + 0.4683w_3)}{|w_0 + 0.4683w_3| + 1} + \frac{e(w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13})}{|w_0 + 0.4124w_{11} + 0.5137w_{12} + 0.5369w_{13}| + 1} + \frac{f(w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23})}{|w_0 + 0.3755w_{21} + 0.4745w_{22} + 0.4911w_{23}| + 1}} - 0.6289 \right)^2 ;
\end{aligned}$$

Finally, the above expression is passed into a minimization function, alongside the list of independent variables. FindMinimum and NMinimize are explored and compared. Also constraints on the independent variables were experimented with.

In[72]:= NMinimize [costFunc, vars, MaxIterations -> 10]

NMinimize::cvmit: Failed to converge to the requested accuracy or precision within 10 iterations. >>

Out[72]= {0.00376706, {e -> -5.61649, f -> 9.72728, g -> 9.02595, k -> 3.30917, w₀ -> -0.552616, w₁ -> 5.03397, w₂ -> -3.62446, w₂₃ -> -3.58699, w₃ -> 1.18056, w₁₁ -> -48.2702, w₁₂ -> 11.7368, w₁₃ -> 16.5787}}

The first number in the above result represents the minimum value of the expression, while the remaining numbers give the values of independent variables at the minimum value. It was however noticed that the minimum value of the expression given depended largely on the number of iterations. The results for 10 and 1000 iterations are given below.

In[73]:= NMinimize [costFunc, vars, MaxIterations → 100]
NMinimize::cvmit: Failed to converge to the requested accuracy or precision within 100 iterations. >>

Out[73]= {0.00101603, {e → -2.28939, f → 2.00419, g → 80.1258, k → -69.3295, w₀ → 3.03287, w₂₁ → 8.59814, w₂₂ → -7.17148, w₂₃ → -6.79922, w₃ → 8.45907, w₁₁ → -36.8971, w₁₂ → 5.6079, w₁₃ → 13.8209}}

In[74]:= NMinimize [costFunc, vars, MaxIterations → 1000]

Out[74]= {0.0000861509, {e → -188.365, f → -164.04, g → 1.15549, k → -346.534, w₀ → -47.5472, w₂₁ → -12 410.2, w₂₂ → 8065.97, w₂₃ → -3499.01, w₃ → 101.552, w₁₁ → -13.7765, w₁₂ → 5.30102, w₁₃ → 10.8999}}

It was found that with 1000 iterations, a much smaller minimum value and hence a somewhat preferable solution was gotten. Substituting the values in the original equation show how closely they match the RHS of the original set of equations. RHS for the original set of equations:

In[75]:= eqns[[All,2]]

Out[75]= {0.3927, 0.3956, 0.3963, 0.3972, 0.4011, 0.4116, 0.4208, 0.4225, 0.4423, 0.4523, 0.4683, 0.6289}

LHS for the original set of equations upon substitution of solution obtained with 10 iterations:

In[76]:= eqns[[All, 1]] /. %72[[2]]

Out[76]= {0.379921, 0.413627, 0.370358, 0.393824, 0.420328, 0.397984, 0.431866, 0.422632, 0.45038, 0.468269, 0.473094, 0.589233}

With 100 iterations:

In[78]:= eqns[[All, 1]] /. %73[[2]]

Out[78]= {0.39043, 0.40142, 0.390105, 0.399946, 0.421613, 0.394592, 0.413064, 0.427783, 0.444282, 0.454655, 0.470642, 0.618022}

In[79]:= eqns[[All, 1]] /. %74[[2]]

Out[79]= {0.392006, 0.400217, 0.393564, 0.396414, 0.403583, 0.415149, 0.416649, 0.420209, 0.439527, 0.45499, 0.468548, 0.628726}

The solution obtained from 1000 iterations is the closest match so far. However, the values of the independent variables seem large; much larger than the desired [-1,1] range. Increasing the number of iterations also doesn't necessarily increase the closeness of the match.

In[80]:= NMinimize [costFunc, vars, MaxIterations → 1 000 000]

Out[80]= {0.0120223, {e → -266 650., f → -11 020.5, g → 328 363., k → -72 735.2, w₀ → 25 874.3, w₂₁ → -1.81561 * 10⁶, w₂₂ → 442 428., w₂₃ → 659 010., w₃ → 50 875.5, w₁₁ → -3149.85, w₁₂ → 1904.91, w₁₃ → 667.346}}

The above solution with 1000000 iterations is even poorer than the solution with 10 iterations.

Minimization with FindMinimum was also explored.

In[81]:= FindMinimum [costFunc, vars, MaxIterations → 10]

FindMinimum::cvmit: Failed to converge to the requested accuracy or precision within 10 iterations. >>

Out[81]= {0.0221383, {e → 0.419081, f → -0.262168, g → 1.75121, k → -0.287054, w₀ → -12.2733, w₂₁ → -33.6519, w₂₂ → -22.7263, w₂₃ → -11.187, w₃ → 35.5723, w₁₁ → -1.93465, w₁₂ → -14.899, w₁₃ → -11.3127}}

In[85]:= FindMinimum [costFunc, vars, MaxIterations → 100]

FindMinimum::cvmit: Failed to converge to the requested accuracy or precision within 100 iterations. .

Out[85]= {0.00614631, {e → 32.7758, f → 29.1231, g → 36.8896, k → 2.67643, w₀ → -0.0553623, w₂₁ → 0.205614, w₂₂ → -0.140589, w₂₃ → -0.165327, w₃ → 0.283651, w₁₁ → 0.274932, w₁₂ → -0.0593741, w₁₃ → -0.0902524}}

In[84]:= FindMinimum [costFunc, vars, MaxIterations → 1000]

FindMinimum::cvmit: Failed to converge to the requested accuracy or precision within 1000 iterations. .

Out[84]= {0.00327674, {e → 2.12202, f → 59.2895, g → 51.0433, k → 48.8821, w₀ → -0.825116, w₂₁ → 0.379462, w₂₂ → -0.298338, w₂₃ → -0.318017, w₃ → 0.515943, w₁₁ → 10.5313, w₁₂ → -2.01802, w₁₃ → -3.49642}}

In[87]:= FindMinimum [costFunc, vars, MaxIterations → 1 000 000]

FindMinimum::sszero:

The step size in the search has become less than the tolerance prescribed by the PrecisionGoal option, but the gradient is larger than the tolerance specified by the AccuracyGoal option. There is a possibility that the method has stalled at a point that is not a local minimum. >>

Out[87]= {0.000692869, {e → 0.00617008, f → 517.749, g → 1.10264, k → 511.352, w₀ → -63.8263, w₂₁ → 0.871495, w₂₂ → 0.0777295, w₂₃ → 0.685556, w₃ → 136.303, w₁₁ → 558.272, w₁₂ → 11.9439, w₁₃ → -288.725}}