A NEW COMPOSITE MULTIDERIVATIVE LINEAR MULTISTEP METHODS FOR SOLVING STIFF INITIAL VALUE PROBLEMS.

by

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CERTIFICATION

THIS IS TO CERTIFY THAT THE

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SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES UNIVERSITY OF LAGOS FOR THE AWARD OF THE DEGREE OF

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IS A RECORD OF ORIGINAL RESEARCH CARRIED OUT BY

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ABSTRACT

Numerical solutions to stiff problems have over the years been a computational problem. Linear Multistep Methods (based on polynomial interpolation) which are popularly used in obtaining solutions to Ordinary Differential Equations have some drawback when applied to stiff Initial Value Problems (IVPs).

A class of Mul iderivative Linear Multistep Methods (MLMMs) of orders 2 to 7 is developed for stiff IVPs. The MLMM is applied to stiff problems in Predictor-Corrector mode. The convergence, consistency and the stability of these methods are inve..tigated and are shown to be suitable for stiff IVPs.

Methods of orders two and four are shown to be accurate and reliable. Numerical comparison of various methods in terms of accuracy and rate of convergence are considered.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 STATEMENT OF PROBLEM

Many physical, biological and management problems giving rise to Ordinary Differential Equations [ODEs] cannot be solved analytically, that is, in closed form.

The focus of interest in this work is the mathematical model

$$\bar{y}' = \bar{f}(x,y), \quad \bar{y}(a) = \bar{\eta} \qquad x \varepsilon [a,b] \qquad (1.1)$$

where $\bar{y} = y(x_i)$, $i = 1,2,\ldots,k$ are known functions of x, a and $\bar{\eta}$ are real known vectors given by

$$\bar{y}(a) = [y_1(a),\ y_2(a), \ldots,\ y_k(a)]^T$$

$$\bar{\eta} = [\eta_1, \eta_2, \ldots, \eta_k]^T$$

There are also some ODEs that do have analytical solutions, but with their numerical results being of greater interest and importance. Such problems therefore require the use of numerical methods for their solutions.

Various numerical approaches for dealing with these situations do exist. One class among such schemes is the discrete variable methods Fatunla [9], Lambert [18]. This class falls into two distinct categories:

a) The One-step or Runge-Kutta (R-K) methods which are essentially substitution methods.

b) The Linear Multistep Methods (LMMs) which are basically polynomial interpolation schemes.

Another group that combined the characteristics of the two distinct classes (a) and (b) is the Hybrid methods.
Numerical investigation of ODEs of the type (1.1) can broadly be carried out for problems regarded as stiff and non-stiff (see Gear [12]).

The R-K methods which has an advantage of being self-starting, has a limitation with its explicit schemes; though an implicit s-stage R-K methods which are A stable can attain order 2s. However for stiff problems, the class of multistep methods has been a class of methods often preferred.

Often most multistep methods are of Adam's type which are broadly divided into Adam-Moulton and Adam-Bashforth Methods (see Lambert [18]). In stiff problems, it is observed that stiffness is a property of mathematical problems and not of the numerical method. Generally, stiff equations are problems for which a typical solution is a rapidly decaying exponentially, Lambert [19]. Their investigation numerically are very tedious. For this reason, there has been serious research attention focussed on this class of problems since Gear introduced the code DIFSUB in 1971 [26].

Hence the purpose of this study is to develop a very efficient numerical method for the solutions of stiff problems, an area which is still open for research, Fatunla [9] and Gear [13].

The approach employed in obtaining the solution of (1.1) is to derive a new version of multistep methods. The usual Linear Multistep Method (LMM) is extended to the case of Multiderivative Linear Multistep Method (MLMM). The MLMM is a scheme which has been considered for the solution of stiff IVPs by various authors. Enright [7], Brown [23] and Twizel and Khaliq [27] have considered the MLMM in various forms as a class of methods for the solution of stiff IVPs. It will be shown that the new set of formulas proposed gives accurate result comparable to other known methods.

The work done in the derivation of the formulas proposed usually involve solving linear systems to determine the parameters and constants governing the formula. The variation allowed in the formulas especially the introduction of free parameters help in the accuracy of the method.

Like some other methods, the system (1.1) possesses a Jacobian matrix J(x) involving partial derivative $\partial f/\partial y$ evaluated at $(x,y(x))$. The case study of this work followed from the work done by Cash [2] and Twizel and Khaliq [27,28]. They fitted an exponential function into a test problem

$$y' = \lambda y \quad , \qquad y(x_0) = y_0 \tag{1.2}$$

and a general second derivative linear k-step method was used for the case k = 1. However, since the method is based on a second derivative formula, the case k = 2 is of a paramount interest.

We also propose a possible extension of the formula with modifications to fitting it to functions other than exponential function.

Thus, appropriate modifications which improve on the efficiency of the method for stiff problems are then carried out having ensured that necessary convergence, consistency and stability conditions are satisfied. The more efficient and reliable algorithm is then coded and implemented on the digital computer.

The idea of integrating trapezoidal rule into the newly developed scheme for stiff system will be considered in chapter three.

Gear [12,13] stated that trapezoidal rule applied to stiff system; gives slowly damped oscillatory errors. However, errors are quickly damped for backward Euler scheme. The magnitude of the error in the trapezoidal rule is controlled by step length choice and the result will still be acceptable.

1.2 NUMERICAL STABILITY FOR STIFF EQUATIONS

The basic problem usually encountered when attempting to obtain numerical approximation to the solution $y(x)$ of stiff equation of the form (1.1) is that of numerical stability. The step length h used in a numerical method to obtain a solution usually contributes to the stability limitation of a method. Dalquist [3,5] investigated the special stability problem connected with stiff equations and introduced the concept of A-Stability.

DEFINITION 1.1 (A-Stability) (Lambert, 1973)
A numerical method is said to be A-stable if the region of absolute stability contains the whole of the left half-plane $Re(\lambda h) < 0$.

Thus to overcome this stability limitation on the step size, h, numerical methods for solution of stiff problems have been sought to possess Region of Absolute Stability (RAS) in the open left half-plane $Re(\lambda h) < 0$. However, in areas of physical importance, that gave rise to system of ODEs, it is often noticed that such system of equations are inherently very stable, but numerical methods often used are impractical because of severe step-size restriction imposed by the requirement of absolute stability.

This sort of difficulty was first encountered in 1951 by Curtis and Hirschfelder [24] in equations governing masses controlled by springs of different stiffness and it later became known as stiff equation.

Consider a test problem

$$\underline{y}' = \underline{A}\ \underline{y} \quad , \quad \underline{y}(x_0) = \underline{y}_0 \tag{1.3}$$

where $\underline{y} = (y_1, y_2, \ldots, y_n)^T$, $y_j \in R$, $\underline{A}$ is $n \times n$ real matrix having eigenvalues λ_j , $j = 1, \ldots, n$ contained in the open left half plane.

Assume that

$$\max_{1 \leq i,j \leq k} \left| \frac{Re\ (\bar{\lambda}_j)}{Re\ (\lambda_i)} \right|$$

is large, then the system of equation (1.3) is inherently stable and the exact solution

$$y(x) = e^{(x-x_o)A}\ y_o \tag{1.4}$$

tends to zero exponentially as x increases.

The difficulty faced with stiff equations is that though the component of the true solution y(x) corresponding to λ_j , the eigenvalues of $\underline{A}$, soon become negligible, the restriction on step size imposed by the numerical stability of standard methods requires that $|\lambda_j h|$ remain small in the range of integration.

Dalquist [3[, Gear [10, 12], Fatunla [9] and some others gave various concepts of stability regions which helped in obtaining methods satisfying some sort of stability criteria in the solutions of stiff problems.

Innconsideration of stiff problems, accuracy is an important factor of solution obtained. Hence one of the stability property given by Gear suggested that systems are solved by methods whose stability region extends in the negative real direction. In order to satisfy good stability criterion and have solution with a very high accuracy, the work in this thesis attempted to introduce some new algorithm based on some of the most recent numerical methods suggested by Cash [2], Fatunla[9], Twizel and Khaliq [28] and Hull and Watt [16].

1.3 LINEAR MULTISTEP METHODS

A Linear Multistep Method (LMM) with step number k for solving (1.1) is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.5}$$

where α_j and β_j are constants and not both α_0 and β_0 are zero, that is, $\alpha_0^2 + \beta_0^2 > 0$

In order to remove the arbitrariness of a constant multiplier on (1.5), we take $\alpha_k = +1$. (1.5) is said to be explicit if $\beta_k = 0$ and implicit if otherwise.

Implicit methods require greater computational efforts but are usually more accurate than the explicit ones for a given step number k, Lambert [18]. Most LMMs are formulated by one of the following approaches:

(a) methods of undetermined coefficients,

(b) numerical differentiation, and

(c) numerical integration.

The last of these is the most efficient for stiff problems. It provides the avenues for error estimation and error analysis which in effect facilitates step-size adjustment, Fatunla [9].

The conventional LMM is essentially seen as a polynomial interpolation procedure, whereby the solution or its derivative is replaced by a polynomial of appropriate degree of the independent variable whose derivative or integral is readily obtained. Clearly, linear k-step methods, k=1,2,... can be derived from various known methods. Such methods of derivation include the Taylor series approach, Newton-Cotes integration formulas and Newton-Gregory interpolation algorithms.

The derivation of other LMMs which may not be included by the approaches described above are found through the determination of the order of such methods, (see Lambert [18]).

1.4 CLASSIFICATION OF LMM

The LMM known to be vast in generating solution to IVP can be classified broadly into explicit and implicit schemes. Before the advent of computer, it was a common practice to express the right hand side of a LMM (1.5) in terms of a power series involving finite difference operators. However, since digital computer is now available, it is more convenient to compute with a fixed LMM and alter the step length, whenever there is demand for greater accuracy.

The existence of characteristic polynomial in LMM has resulted in further classification of LMM of different step numbers which share a common form of the first characteristic polynomial $\rho(\xi)$. Thus, methods for which all the spurious roots of the first characteristic polynomial are located at the origin such that

$$\rho(\xi) = \xi^{k} - \xi^{k-1}$$

are called Adams methods. They are known to be zero stable (Lambert [18]). Adams methods which are explicit are called Adams-Bashforth methods, while the implicit Adams methods are Adams-Moulton.

Comparison of explicit and implicit LMMs reveals that the implicit schemes are more accurate with better stability properties than the explicit ones, for a given step number k. Furthermore, the highest attainable order for a zero stable method is less in the case of an explicit method than an implicit one. The explicit methods are often used for initial evaluation of values especially when implicit scheme is being used.

Thus for higher accuracy, the combination of these two methods is often preferred to serve as Predictor and Corrector (P-C) schemes. The explicit method usually serve as Predictor while the Corrector method is implicit in nature.

For every implementation of P-C methods, three steps are involved. These are given below as follows:

Step 1 Predict the star ting value $y_{n+k}^{[0]}$ by

$$p: \quad y_{n+k}^{[o]} = h \sum_{j=o}^{k-1} \beta_j f_{n+j} - \sum_{j=0}^{k-1} \alpha_j y_{n+j}$$

Step 2 Evaluation of $f_{n+k}^{[r]}$ by

$$E: \quad f_{n+k}^{[r]} = f(x_{n+k}, y_{n+k}^{[r]}), \quad r = 0,1,2,\ldots$$

Step 3 Correction of $y_{n+k}^{[r]}$ by

$$C: \quad y_{n+k}^{[r+1]} = h \beta_k f_{n+k} + \sum_{j=0}^{k-1} (h \beta_j f_{n+j} - \alpha_j y_{n+j})$$

A combination of the three steps is called PEC mode. There are two ways in which this mode can be implemented. This can either be in the mode $P(EC)^m$ or $P(EC)^m E$. for some positive integer m, see Lambert [16].

Generally, it is observed that LMM is limited in generating solution for various classes of IVP, especially stiff problems. The draw back centered on the inability of the Backward Differentiation Formula (BDF) and indeed the LMM to cater for the fast decaying exponential property of stiff problems as discussed in section 1.1. Thus due to this limitation of the LMM, the concept of Multiderivative Linear Multistep Method (MLMM) is being considered for generating solution to stiff problems. Appropriate modifications that improved on the efficiency of this new method for stiff equations are then carried out. This is then coded and implemented on the digital computer.

1.5 MULTIDERIVATIVE FORMULAE

The general Multiderivative Linear Multistep Method (MLMM) is given by the formula

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = \sum_{i=0}^{n} h^i \sum_{j=0}^{k} \tau_{i,j} \, f^{(i)}_{n+j} \qquad (1.6)$$

where $f^{(i)}_{n+j}$ are the i-th derivative of $f(x,y)$ evaluated at (x_{n+j}, y_{n+j}).

The method of interest in this work, in particular, is the second derivative LMM obtained from (1.6) when $n = 2$.

Then we have

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h \sum_{j=0}^{k} \beta_j \, f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j \, g_{n+j} \qquad (1.7)$$

where $g_{n+j} = g(x,y) \mid_{n+j} = \dfrac{df(x,y)}{dx}$

and $\beta_j = \tau_{1,j}$, $\gamma_j = \tau_{2,j}$

$\alpha_k = +1$, and not all γ_j , $j = 0,1,\ldots,k$ are zero satisfying $\sum_{j=0}^{k} \alpha_j^2 > 0$

The attainable order for n^{th}-derivative linear k-step method is given by kn+k+n-1 for implicit methods and kn+k-1 for explicit ones. Attempt was made by Cash [2] to fit exponential functions to the composite form of (1.7) with k = 1 for the test problem

$$y' = \lambda y , \quad y(0) = 1 \quad , q = \lambda h \qquad (1.8)$$

Cash's work showed that the method is A-stable for orders $\leqslant 4$.

However, we shall consider case k=2 of (1.7) so as to have higher orders leading to greater accuracy. This shall require greater computational work than the one step case. However it will exhibit a much better accuracy and stability comparable to the analytical results. It will be observed that MLMM exhibits some better accuracy than the usual LMM. This is partly due to higher powers of h introduced in (1.6) for the former method.

Due to the nature of the MLMMs in (1.7), they require greater numerical and analytical work than the LMMs. However, the derivation of these methods reduces the amount of work required on the digital computer and still retain the stability properties for the case $q \to 0$ and $q \to -\infty$ when tested on the scalar problem (1.8). The set of formula to be considered is a pair in the Predictor-Corrector form based on the second derivative LMM of (1.7).

The characteristic polynomials of any of the form (1.7) are given by

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$$

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$$

$$\theta(\xi) = \sum_{j=0}^{k} \gamma_j \xi^j$$

The stability region of the MLMM can be determined by observing that on the boundary R, one of the roots of the polynomial

$$\rho(\xi) - q\sigma(\xi) - q^2\theta(\xi) = 0 \qquad (1.9)$$

will have modulus one.

Hence, when MLMM is fitted to scalar problem (1.8) with $q = \lambda h$, then the Region of Absolute Stability (RAS) in the q plane of the MLMM is that region for which the roots of the characteristic polynomial

$$\sum_{j=0}^{k} (\alpha_j - \beta_j q - \gamma_j q^2)$$

lie in the open unit circle.

CHAPTER TWO

TWO-STEP INTEGRATION FORMULAS

2.1 INTRODUCTION

Linear Multistep Method (LMM) whose derivation is based on polynomial interpolation, given by equation (1.5) performs poorly when applied to stiff Initial Value Problems (IVPs). Hence a form of the LMM which is an Implicit Backward Differentiation Formula (BDF) of (1.5), that is

$$\sum_{j=0}^{k} \alpha_j y_{n+j} - h \beta_k f_{n+k} = 0 \qquad \alpha_k = 1 \tag{2.1}$$

was proved to satisfy the definition of stiff stability by Gear [10]. This was however seen to be less efficient for exponentially fitted problems.

The following definitions are necessary for the derivation of exponentially fitted formulas which satisfy the stiff stability criteria.

DEFINITION 2.1 (Stiff Stability) **Gear (1971)**

A numerical Integrator is stiffly stable if its stability region contains a region of the form $R_1 \cup R_2$, where

$$R_1 = \{ \mathrm{Re}(z) \leq D < 0 \}$$

$$R_2 = \{ D < \mathrm{Re}(z) < \delta \;, \; | \mathrm{Im}(z) | < \theta \;, \; \delta > 0 \}$$

DEFINITION 2.2 (Exponential Fitting) Lambert (1973)

A numerical method is said to be exponentially fitted at a (complex) value λ_0, if when the method is applied to the scalar test problem $y' = \lambda y$, $y(x_0) = y_0$, with exact initial conditions, it yields the exact theoretical solution for $\lambda = \lambda_0$

The stiffly stable LMM requires that the method be implicit. This has led to several investigation into possible ways of stabilising explicit methods by using Jacobian matrix. One means by which this was done is the application of Jacobian matrix on generalizing LMM by calculating the second derivative y''. This led to the use of second derivative multistep formulae. This kind of methods is implemented using free parameters to allow for exponential fitting. Enright [7] developed a class of stiffly stable k-step second derivative methods of order k+2.

2.2 EXPONENTIALLY FITTED FORMULA

The aim in this thesis is to derive exponentially fitted formulae of various orders for which stability requirements are investigated for all choices of the fitting parameters. The idea of using exponentially fitted formulae for stiff problem in the form (2.1) was first proposed by Liniger and Willoughby [20]. Integration formulae containing free parameters were derived and these parameters were chosen so that a given function exp(q) where q is real, satisfies the integration formula exactly. This was tested on LMM for k=1, however Jackson and Kenue [17] have derived a fourth order exponentially fitted formulae based on a linear 2-step formula and were A-stable. These methods which have been cited are deficient in the sense that they do not give estimate of the local truncation error of the formula being used. This makes control of the step length h being used difficult to choose and can lead to gross inefficiencies if an appropriate value of h is not used. The new methods being described in this thesis contain a 'built-in' local error estimate. This error estimate may be used as a basis of a step control procedure, and the resulting variable step formulae are particularly efficient in the transcient phase of the region of integration.

Based on this idea, Cash [2], in his own work, attempted using MLMM with k=1 in the second derivative formulae. We shall however derive a more difficult exponentially fitted formulas for the case k = 2 of the MLMMs using a composite formula. The class of methods to be derived, using the composite MLMMs require the maximum attainable order for the case k=2, n=2 to be seven. Hence various formulas of orders ranging from 2 to 7 will be given.

A composite formula is obtained by applying the simple formula exhibited , repeatedly to cover longer intervals.

The methods being proposed here is basically for stiff problems for which exponential fitting is appropriate. These are often recognised through some knowledge of the physical behaviour of the solution. This procedure is preferred because it is usually found that exponentially fitted integration formulae are substantially more efficient than conventional ones. A numerical investigation of the proposed methods will lead to some conclusion on the stability properties of the methods. The integration formulas which shall be used for the exponential fitting are as follows: The predictor method is chosen to coincide with the second derivative LMM (1.7), given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j} \qquad (2.1)$$

while the corrector formula which is

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k+1} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j} \qquad (2.3)$$

where $g_{n+j} = \left.\frac{df(x,y)}{dx}\right|_{x}^{y}$ and $\beta_{k+1} = 0$.

For this purpose of deriving effective schemes of different orders, the following definition is given.

Define the L operator as

$$L[y(x;h] = \sum_{j=0}^{k} \{ \alpha_j y(x+jh) - h\beta_j f(x+jh, y(x+jh)]$$
$$- h^2 \gamma_j g(x+jh, y(x+jh)]$$

and by using Taylor series expansion and collecting like terms, we can write L as

$$L = c_0 y(x) + hc_1 y'(x) + \ldots + h^r c_r y^{(r)}(x)$$

where

$$c_o = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_K$$

$$c_1 = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k - (\beta_0 + \beta_1 + \ldots + \beta_K)$$

$$c_2 = \tfrac{1}{2}(\alpha_1 + 2^2\alpha_2 + \ldots + k^2\alpha_k) - (\beta_1 + 2\beta_2 + \ldots + k\beta_k)$$

$$c_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \ldots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots$$

$$k^{q-1}\beta_k) - \frac{1}{(q-2)!}(\gamma_1 + 2^{q-2}\gamma_2 + \ldots + k^{q-2}\gamma_k)$$

$$q = 3,4,\ldots, r$$

The Method (2.2) or (2.3) is of order p if

$$c_0 = c_1 = \ldots = c_p = 0 \quad \text{and} \quad c_{p+1} \neq 0$$

Thus to derive a scheme of order p, we set $c_j = 0$, $j = 0,1,\ldots,p$ giving rise to sets of (p+1) equations from which constant parameters α_j, β_j and γ_j are determined according to equation (2.2) or (2.3).

The algorithm required for the study of the stability criterion connects the Predictor and the Corrector formulas together and is given as follows:

A1 Compute $\bar{y}_{n+k}$ as solution of (2.2)

A2 Compute $\bar{f}_{n+k} = f(x_{n+k}, y_{n+k})$ and $\bar{g}_{n+k} = g(x_{n+k}, y_{n+k})$

A3 Compute $\bar{y}_{n+k+1}$ as the solution of,

$$\sum_{j=0}^{k-1}[\alpha_j y_{n+j+1} - h\beta_j f_{n+j+1} - h^2\gamma_j g_{n+j+1}]$$

$$= \alpha_k y_{n+k+1} + h\beta_k f_{n+k+1} + h^2\gamma_k g_{n+k+1}$$

A4 Compute $f_{n+k+1} = f(x_{n+k+1}, y(x_{n+k+1}))$

A5 Compute y_{n+k} as the solution of

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j} + h \beta_{k+1} f_{n+k+1}$$

This algorithm shall henceforth be regarded as Composite Integration Formula (CIF) and the complete scheme defined will be referred to as steps A1 - A5.

2.3 DERIVATION OF ORDER 2 SCHEME

The first scheme to be considered is of order 2. The procedure employed in this thesis requires that the Predictor formula be of order (k-1) while the Corrector scheme is of order k.

The predictor formula corresponding to the case k=2, with one free parameter is obtained from equation (2.2) as

$$\begin{aligned} \alpha_0 y_n + \alpha_1 y_{n+1} + y_{n+2} &= h (\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}) \\ &+ h^2 (\gamma_0 g_n + \gamma_1 g_{n+1} + \gamma_2 g_{n+2}) \end{aligned} \qquad (2.5)$$

Hence, to derive a two step scheme of order 2, set coefficients of y, f and g at x_{n+1} point to zero, that is

$\alpha_1 = \beta_1 = \gamma_1 = 0$

Also let $\gamma_0 = \gamma_2 = 0$, then we have α_0, α_2, β_0 and β_2 to determine.

But $\alpha_k = \alpha_2 = +1$ and let $\beta_2 = a$ (free parameter).

For order two predictor formula set $c_0 = c_1 = 0$; $c_2 \neq 0$

using equation (2.4) we obtain $\alpha_0 = -1$, $\beta_0 = 2-a$

Thus (2.5) becomes

$$y_{n+2} - y_n = h [a f_{n+2} + (2-a) f_n]$$

but $y' = f(x,y)$

that is $y_{n+2} - y_n = h [a y'_{n+2} + (2-a) y'_n]$

By exponential fitting, we use equation (1.8) to get step A1 of the Algorithm as

$$\bar{y}_{n+2} - y_n = h\,[a\lambda\,\bar{y}_{n+2} + (2-a)\lambda\, y_n\,]$$

leading to

$$(1-aq)\,\bar{y}_{n+2} = [1 + (2-a)q\,]\,y_n$$

or
$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1+(2-a)q}{1-aq} \tag{2.6}$$

where $q = \lambda h$

From where step A2 of the CIF is also given.

The solution of the scalar problem (1.8), that is

$$y' = \lambda y \quad \text{is} \quad y = e^{\lambda x}$$

$$\frac{y_{n+1}}{y_n} = \frac{y(x_n+h)}{y(x_n)} = \frac{e^{\lambda(x_n+h)}}{e^{\lambda x}} = e^{\lambda h} = e^q$$

similarly,
$$\frac{y_{n+2}}{y_n} = e^{2q} \tag{2.7}$$

using (2.7) in (2.6), we obtain

$$e^{2q} = \frac{1+(2-a)q}{1-aq}$$

solving for parameter a, we get

$$a = \frac{(1-e^{2q})/q + 2}{1-e^{2q}} \tag{2.8}$$

From equation (2.3), the corrector formula corresponding for k=2 is

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h\,(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3})$$
$$+ h^2\,(\gamma_0 g_n + \gamma_1 g_{n+1} + \gamma_2 g_{n+2})$$

Again, setting $\alpha_1 = \beta_1 = 0$, $\gamma_0 = \gamma_1 = \gamma_2 = 0$, $\alpha_2 = +1$ and $\beta_3 = r$ (free parameter), then a method of order 2 involving corrector formula requires

$$c_0 = c_1 = c_2 = 0 \quad \text{and} \quad c_3 \neq 0$$

This leads to the simultaneous equations

$$\alpha_0 + 1 = 0$$

$$2 - \beta_0 - \beta_2 - r = 0$$

$$2 - 2\beta_2 - 3r = 0$$

which give

$$\alpha_0 = 1, \; \beta_0 = \tfrac{1}{2}(2+r), \; \beta_2 = \tfrac{1}{2}(2-3r)$$

hence the method becomes

$$y_{n+2} - y_n = h \left[r y'_{n+3} + \tfrac{1}{2}(2-3r) y'_{n+2} + \tfrac{1}{2}(2+r) y'_n \right] \quad (2.9)$$

solving for r to get

$$r = \frac{(e^{2q} - 1)/p - e^{2q} - 1}{e^{2q} - \frac{3}{2} e^{2q} + \frac{1}{2}} \quad (2.10)$$

Proceeding on step A3 of the CIF we should have

$$\alpha_0 y_{n+1} + \alpha_1 y_{n+2} + \alpha_2 y_{n+3} - h\,(\beta_0 f_{n+1} + \beta_1 f_{n+2} + \beta_2 f_{n+3})$$
$$- h^2\,(\gamma_0 g_{n+1} + \gamma_1 g_{n+2} + \gamma_2 g_{n+3}) = 0$$

This equation is a transpose of equation (2.5) by a step, hence if we solve for the constants α_j , β_j and γ_j with the same condition impose on (2.5), we shall obtain

$$y_{n+3} - y_{n+1} - h(2-a)f_{n+1} - ahf_{n+3} = 0$$

Applying exponential fitting condition to get

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \frac{1 + (2-a)q}{1 - aq} \quad (2.11)$$

This result is identical to the predictor formula (2.6).

Hence following this trend, we observe that

$$\frac{\bar{y}_{n+j+2}}{y_{n+j}} = \frac{1 + (2-a)q}{1 - aq}, \quad j = 0,1,\ldots,r$$

satisfying the consistency conditions.

There is the need to obtain ratio $\dfrac{\bar{y}_{n+3}}{y_n}$ in order to compute step A4 of the CIF. Thus from (2.11), using (2.7) we have

$$\frac{\bar{y}_{n+3}}{y_n} = \frac{1 + (2-a)q}{1 - aq} \cdot \frac{y_{n+1}}{y_n} = e^{3q}$$

that is

$$\frac{\bar{y}_{n+3}}{y_n} = \left(\frac{1 + (2-a)q}{1 - aq} \right)^{\frac{3}{2}} = R^*(q) \qquad (2.12)$$

Finally for step A5, equation (2.9) becomes

$$\frac{y_{n+2}}{y_n} \quad \frac{rq\ (\bar{y}_{n+3}/y_n) + 1 + \frac{1}{2}q(2+r)}{1 - \frac{1}{2}q(2-3r)}$$

substituting (2.12) to get

$$\frac{y_{n+2}}{y_n} = \frac{rq\ R^*(q) + 1 + \frac{1}{2}q(2+r)}{1 - \frac{1}{2}q(2-3r)} = R(q) \qquad (2.13)$$

This unites both the predictor and the corrector formulas. The formula (2.13) derived is capable of generating solution to stiff problems for which exponential fitting is applicable.

A good numerical method must satisfy reasonable stability properties. It is therefore necessary to find the range of values of a and r with certain limits of q.
To determine the conditions a and r will satisfy, we examine the inequality

$$\left| \frac{y_{n+2}}{y_n} \right| < 1$$

for all q with $Re(q) < 0$

The necessary and sufficient conditions for this inequality to hold are given by Maximum Modulus Theorem.

THEOREM 2.1 (Maximum Modulus Theorem)

Let f be analytic and not constant in a domain D. Then $|f|$ cannot have a local maximum in D.

The proof is given in Beardon [1]

By Theorem 2.1, these conditions imply that

(i) $R(q) < 1$ on $Re(q) = 0$

(ii) $R(q)$ is analytic in $Re(q) < 0$

If (i) holds, it follows that R(q) is analytic as $q \to -\infty$ and thus (i) and (ii) will guarantee A-stability by Theorem 2.1.

Now, for $|R(q)| < 1$, we consider

$$\frac{y_{n+2}}{y_n} - 1 < 0$$

hence from (2.13) we have

$$\frac{\left(\dfrac{1/q + (2-a)}{1/q - a}\right)^{\frac{3}{2}} + \frac{1}{q} + \frac{1}{2}(2+r) - \frac{1}{q} + \frac{1}{2}(2-3r)}{1/q - \frac{1}{2}(2-3r)} < 0$$

and as $q \to -\infty$, we obtain

$$\frac{r(1-\frac{2}{a})^{\frac{3}{2}} + 2 - r}{\frac{3}{2}r - 1} < 0$$

leading to $r < \frac{2}{3}$ and $a > 0$.

These inequalities may help to determine stability condition. However, by taking limits of a and r as $q \to 0$ and $q \to -\infty$, we obtain from (2.8) and (2.10) that

$$\lim_{q \to 0} a = \lim_{q \to 0} \frac{1 - e^{2q} + 2q}{q(1 - e^{2q})} = 1$$

Also,
$$\lim_{q \to -\infty} a = \lim_{q \to -\infty} \frac{(1 - e^{2q})/q + 2}{1 - e^{2q}}$$

Similarly,
$$\lim_{q \to 0} r = -\frac{4}{9}, \quad \text{while} \quad \lim_{q \to -\infty} r = -2$$

Taking values of $q \in (-\infty, 0)$ for a large sample S, we observed that for various values of q, the values of a and r are within the range above, that is $a \in (1,2)$ and $r \in (-2, -\frac{4}{9})$

Thus for such sample we have table 2.1 below.

TABLE 2.1
Values of parameters a and r for order 2 scheme

q	a	r
-1	1.3130	-0.7850
-2	1.5373	-1.1105
-5	1.8001	-1.6003
-50	1.9800	-1.9600
-100	1.9900	-1.9800
-200	1.9980	-1.9960

This set of values suggests that our integration formula may be A-stable because according to theorem 2.1, all the values of a and r within these ranges are bounded.

On the other hand, if equation (1.9) is applied to the predictor formula, we have

$$\xi^2 = \frac{1 + q(2-a)}{1 - aq} = \frac{y_{n+2}}{y_n}$$

however, since R(q) gives the stability region as $a > 0$ and $r < \frac{2}{3}$ then for $0 < a < 1$, $q = -0.125$, zero stability is satisfied since $|\xi| < 1$ for all a.

Furthemore, the combined Predictor-Corrector formula (2.13), substituted into (1.9) gives

$$\xi^2 = \frac{rq\left(\frac{1 + (2-a)q}{1 - aq}\right)^{\frac{3}{2}} + 1 + \frac{1}{2}q(2+r)}{1 - \frac{1}{2}q(2-3r)}$$

thus, for $r < \frac{2}{3}$ and $a > 0$, we have for the range $a \in (1,2)$ and $r \in (-0.78, 1.99)$ that $|\xi| < 1$;

for example, when $a = 1.8$, $r = -1.6$ we obtain a complex value

$$\xi = -0.187i + 0.684$$

for which $|\xi| = 0.709 < 1$

This is true for all value pairs of a and r, hence the method is absolutely stable for all choices of free parameters a and r of the Predictor-Corrector method (2.13).

2.4 DERIVATION OF ORDER 3 SCHEME

The derivation of Predictor-Corrector integration formula of order 3 , for the class of MLMM being used varies. This is due to the maximum order required , thereby limiting the number of equations available for the determination of coefficients in equation (2.2) and (2.3).

For the Predictor formula given by (2.2), set $\alpha_0 = 1$, $\alpha_1 = \beta_1 = \gamma_1 = 0$ and let $\beta_2 = a$ (free parameter).

Fitting the Predictor formula to second order formula, we obtain,

$$\alpha_0 = -1$$

$$\beta_0 = 2 - a$$

and $\gamma_0 + \gamma_2 = 2 - 2a$

These lead to the choice

$$\gamma_0 = \gamma_2 = 1 - a$$

therefore (2.2) becomes

$$y_{n+2} - y_n = h\,[(2-a)f_n + af_{n+2}] + h^2[(1-a)g_n + (1-a)g_{n+2}]$$

$$\Rightarrow \quad \frac{y_{n+2}}{y_n} = \frac{1 + (2-a)q + (1-a)q^2}{1 - aq - (1-a)q^2}$$

$$= \frac{(q-1)[(a-1)q - 1]}{(q+1)[(a-1)q + 1]} = R^*(q) \qquad (2.14)$$

when a=1, (2.14) may not satisfy the stability requirements because

$$\lim_{q \to -\infty} \left| \frac{y_{n+2}}{y_n} \right| = 1$$

However, by choosing

$$\gamma_0 = 1 - \tfrac{1}{2}a \quad \text{and} \quad \gamma_2 = 1 - \tfrac{3}{2}a$$

together with other coefficients obtained, we have equation (2.2) becoming

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (1-\frac{1}{2}a)q^2}{1 + aq - (1-\frac{3}{2}a)q^2} = \bar{R}\,(q) \qquad (2.15)$$

evaluating a from this we shall obtain

$$a = \frac{(1-e^{2q})/q + 2 + q(e^{2q} + 1)}{1 - e^{2q} + \frac{1}{2}q(1+3e^{2q})} \qquad (2.16)$$

For the composite Corrector formula for case k=2 from equation (2.3) we set

$$c_0 = c_1 = c_2 = c_3 = 0$$

since $\alpha_2 = 1$, $\beta_3 = r$ (free parameter)

then, for $\alpha_1 = \beta_1 = \gamma_1 = 0$, we solve system of equation (2.4) to get

$$\alpha_0 = -1, \quad \beta_0 + \beta_2 = 2 - r$$

$$\gamma_0 = \frac{1}{3} - \frac{25}{4}\, r\,, \quad \gamma_2 = -\frac{1}{3} + \frac{5}{4}\, r$$

choosing $\beta_0 = 1 - 2r$ and $\beta_2 = 1 + r$

then (2.3) becomes

$$y_{n+2} - y_n = h[(1-2r)f_n + (1+r)f_{n+2} + rf_{n+3}] + h^2[(\tfrac{1}{3} - \tfrac{25}{4}r)g_n + (-\tfrac{1}{3} + \tfrac{5}{4}r)g_{n+2}]$$

using exponential fitting to get

$$r = \frac{(e^{2q} - 1)/q - (1 + e^{2q}) + \frac{1}{3}q(e^{2q} - 1)}{e^{3q} + \frac{1}{4}(4+5q)e^{2q} - \frac{1}{4}(8 + 15q)} \tag{2.17}$$

and

$$\frac{y_{n+2}}{y_n} = \frac{4rqR^*(q) - rq(8+15q) + 4(1+q+\frac{1}{3}q^2)}{4(1-q+\frac{1}{3}q^2) - rq(4+5q)}$$

$$= \bar{R}(q). \quad \text{say} \tag{2.18}$$

where

$$R^*(q) = \frac{\bar{y}_{n+3}}{y_n} = \left(\frac{\bar{y}_{n+2}}{y_n}\right)^{\frac{3}{2}} = [\ \bar{R}(q)\ ]^{\frac{3}{2}}$$

is given by step A3 of the CIF.

To examine the stability conditions required by this method, it is expected by Theorem 2.1 that (2.18) satisfies $|\bar{R}(q)| < 1$. However, since equation (2.15) is contained in (2.18), then (2.15) must also satisfy $|\bar{R}(q)| < 1$.

From (2.18), $R(q) - 1 < 0$, as $q \to -\infty$ gives

$$\frac{\frac{4r}{q} R^*(q) - 10r}{\frac{4}{3} - 5r} < 0$$

that is, $r < \frac{4}{15}$.

Furthermore, from (2.15) examine

$$\bar{R}(q) - 1 < 0 \quad \text{as} \quad q \to -\infty$$

and after tediuos algebra, we obtain

$$\frac{2 - 2a}{-1 + \frac{3}{2}a} < 0$$

$\Rightarrow \qquad a < \frac{2}{3} \quad \text{or} \quad a > 1$

Also, taking $|R(q)| < 1$ as $-1 < R(q) < 1$,

then for $R(q) > -1$ we have $a > \frac{2}{3}$ or $a < 0$.

However from (2.16) the range of values of a for $q \in (-\infty, 0)$ taking the limits with L'Hospital rule we have

$$\lim_{q \to 0} a = \frac{2}{3} \quad \text{while} \quad \lim_{q \to -\infty} a = 2$$

that is, $a \in (\frac{2}{3}, 2)$.

Thus, as q decreases, a is monotone increasing.

Also, from (2.17)

$$\lim_{q \to 0} r = 1 \quad \text{and} \quad \lim_{q \to -\infty} r = \frac{4}{45}$$

That is for all values of $q < 0$ parameters r lies in the interval $(\frac{4}{45}, 1)$.

Thus as q increases, r also is monotone increasing.

For the ranges of a and r obtained, they suggest that the integration formula (2.18) may be A-stable. However, we established from algebraic manipulation on the condition that $R(q)$ satisfy with ranges of values of a and r, that is within the ranges

$$a \in (1,2) \quad \text{and} \quad r \in (\tfrac{4}{45}, \tfrac{4}{15})$$

our integration formula (2.18) will satisfy the A-stability conditions given by the Maximum Modulus Theorem 2.1.

Alternative to the order three formulas (2.15) and (2.18) obtained above is by introducing two free parameters. The reason being that the choice of γ_0 and γ_2 for the predictor formula (2.15) and the choice made for β_0 and β_2 as it affects the corrector formula will go a long way to affect the stability and the accuracy of the Predictor-Corrector scheme (2.18). This will also affect the free parameters a and r. Thus by introducing two free parameters each into the predictor and the corrector formulas, a new set of formulas are obtained.

For the Predictor formula, set

$$\alpha_1 = \beta_1 = \gamma_1 = 0 , \quad \alpha_2 = +1$$

and let $\beta_2 = a$ and $\gamma_2 = b$ (free parameters)

then, $\beta_0 = 2 - a$, $\alpha_0 = -1$

$$\gamma_0 = 2 - 2a - b$$

which gives the Predictor formula as

$$y_{n+2} - y_n = h\,[(2-a)f_n + af_{n+2}] + h[(2-2a-b)g_n + bg_{n+2}]$$

using exponential fittings we have

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (2-2a-b)q^2}{1 - aq - bq^2} = \bar{R}\,(q) \qquad (2.19)$$

In a similar manner, for corrector formula, set

$\beta_3 = r$, $\beta_2 = s$ as free parameters;

then for an order 3 corrector scheme in form of (2.3) we have

$$y_{n+2} - y_n = h[(2-s-r)f_n + sf_{n+2} + rf_{n+3}]$$
$$+ h^2[(\tfrac{4}{3} - s - \tfrac{3}{4}r)g_n + (\tfrac{2}{3} - s - \tfrac{9}{4}r)g_{n+2}]$$

on using (1.8) we obtain

$$\frac{y_{n+2}}{y_n} = \frac{1 + (2-s-r)q + (\frac{4}{3} - s - \frac{3}{4}r)q + rq\,(y_{n+3}/y_n)}{1 - sq - q(\frac{2}{3} - s - \frac{9}{4}r)} \qquad (2,20)$$

By step A3 of the Algorithm, we obtain

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \bar{R}\,(q) = \frac{y_{n+2}}{y_n}$$

or $$\frac{\bar{y}_{n+3}}{y_n} = \left(\frac{y_{n+2}}{y_n}\right)^{\frac{3}{2}} = R^*(q)$$

making corrector formula become

$$\frac{y_{n+2}}{y_n} = \frac{1 + (2-s-r)q + (\frac{4}{3} - s - \frac{3}{4}r)q^2 + rq.R^*(q)}{1 - sq - q(\frac{2}{3} - s - \frac{9}{4}r)}$$

$$= R\ (q) \qquad (2.21)$$

Now using exponential fitting on (2.19), we obtain

$$(e^{2q} - 1) - 2q(q+1) = aq(e^{2q} - 1 - 2q) + bq^2(e^{2q} - 1) \qquad (2.22)$$

For the two unknowns a and b, this formula is fitted to two values of q as q_0 and q_1.

Thus let
$$C_i = e^{2q} - 1 - 2q_i(q_i+1)$$
$$A_i = q_i(e^{2q} - 1 - 2q_i)$$
$$B_i = q_i^2(e^{2q} - 1)$$

then (2.22) can be written as

$$C_i = aA_i + bB_i \qquad i = 0,1$$

leading to two equations

$$C_0 = aA_0 + bB_0$$

$$C_1 = aA_1 + bB_1$$

in two unknowns. Solving for a and b we obtain

$$a = \frac{B_1C_0 - B_0C_1}{B_1A_0 - B_0A_1}$$

$$b = \frac{A_1C_0 - A_0C_1}{B_1A_0 - B_0A_1}$$

Similarly, the corrector formula (2.20) can be written as

$$e^{2q}(1 - \tfrac{2}{3}q^2) - 1 - 2q - \frac{4}{3}q^2 = s[q(e^{2q} - 1) - q^2(e^{2q} + 1)]$$

$$+ r[q(e^{3q} - 1) - \frac{3}{4}q^2(3e^{2q} + 1)$$

If $T_i = e^{2q}(1 - \tfrac{2}{3}q_i^2) - 1 - 2q - \frac{4}{3}q_i^2$

$$S_i = q_i(e^{2q} - 1) - q_i(e^{2q} + 1)$$

$$R_i = q_i(e^{3q} - 1) - \frac{3}{4}q_i^2(3e^{2q} + 1)$$

then

$$T_i = sS_i + rR_i \qquad i = 0,1$$

or

$$T_0 = sS_0 + rR_0$$

$$T_1 = sS_1 + rR_1$$

solving to get

$$s = \frac{R_1T_0 - R_0T_1}{R_1S_0 - R_0S_1} \qquad \text{and} \qquad r = \frac{S_1T_0 - S_0T_1}{S_1R_0 - S_0R_1}$$

To investigate the behaviour of the parameters a, b, s and r, substitute (2.21) into the inequality

$$R(q) - 1 < 0$$

After much algebra we shall obtain

$$s > -\frac{1}{6} \qquad \text{and} \qquad s + \frac{9}{4}r - \frac{2}{3} > 0$$

with $R^*(q) < 1$ which by (2.19) we have

$$\tilde{R}(q) < 1 \qquad \text{or} \qquad \frac{2(a-1)}{b} < 0$$

If $b > 0$, then $a > 1$.

For all real roots of the quadratic expression of the numerator of (2.19), we have

$$(a+2)^2 > 4(2-b)$$

and its denominator gives

$$a^2 + 4b > 0 \Rightarrow \quad b > 0$$

In conclusion, the behaviour of these parameters along with the analytical conclusion above is best done by keeping q_1 fixed and q_0 is varied in the interval $(-\infty, 0]$ to satisfy stiff problem (1.8).

2.5 DERIVATION OF ORDER 4 SCHEME

Deriving a second derivative Predictor formula of order 3 corresponding to (2.5) which also contains a free parameter, deduce from (2.4) that

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + 2\alpha_2 - \beta_0 - \beta_1 - \beta_2 = 0$$

$$\frac{1}{4}(\alpha_1 + 4\alpha_2) - (\beta_1 + 2\beta_2) - (\gamma_0 + \gamma_1 + \gamma_2) = 0$$

$$\frac{1}{6}(\alpha_1 + 8\alpha_2) - \tfrac{1}{2}(\beta_1 + 4\beta_2) - (\gamma_1 + 2\gamma_2) = 0$$

Obtaining a two step formula, we set $\alpha_1 = \beta_1 = \gamma_1 = 0$ and let $\beta_2 = a$ (free parameter), then

$\alpha_2 = 1$, $\alpha_0 = -1$, $\beta_0 = 2 - a$, $\gamma_0 = \frac{4}{3} - a$, and $\gamma_2 = \frac{2}{3} - a$

Thus the predictor method is

$$y_{n+2} - y_n = h[(2-a)y_n' + ay_{n+2}'] + h^2[(\tfrac{4}{3} - a)y_n'' + (\tfrac{2}{3}-a)y_{n+2}''] \qquad (2.24)$$

Using (1.8), we have

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (\frac{4}{3} - a)q^2}{1 - aq - (\frac{2}{3} - a)q^2} = \bar{R}(q) \qquad (2.25)$$

and by (1.8) and (2.7), equation (2.24) can be written as

$$a = \frac{1 + 2q + \frac{4}{3}q^2 + \frac{1}{3}e^{2q}(2q^2 - 3)}{qe^{2q}(q-1) + q(q+1)} \qquad (2.26)$$

The Corrector formula of order 4 corresponding to (2.3) gives the set of coefficient equations to be determined as follows:

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + 2\ \alpha_2 - \beta_0 - \beta_1 - \beta_2 - \beta_3 = 0$$

$$\tfrac{1}{2}\ (\ \alpha_1 + 4\ \alpha_2) - (\ \beta_1 + \ \beta_2 + 3\ \beta_3) - (\ \gamma_0 + \ \gamma_1 + \gamma_2) = 0$$

$$\frac{1}{6}(\ \alpha_1 + 8\ \alpha_2) - \tfrac{1}{2}\ (\ \beta_1 + 4\ \beta_2 + 9\ \beta_3) - (\ \gamma_1 + 2\ \gamma_2) = 0$$

$$\frac{1}{24}(\ \alpha_1 + 16\ \alpha_2) - \frac{1}{6}(\ \beta_1 + 8\ \beta_2 + 27\ \beta_3) - \tfrac{1}{2}\ (\ \gamma_1 + 4\ \gamma_2) = 0$$

setting $\alpha_1 = \beta_1 = \gamma_1 = 0$, $\alpha_2 = 1$ and the free parameter $\beta_3 = r$ we obtain other coefficients as

$$\alpha_0 = -1, \quad \beta_0 = 1-r\ , \quad \beta_2 = 1, \quad \gamma_0 = \frac{1}{3} - \frac{3}{4}\ r \quad \text{and} \quad \gamma_2 = -\frac{1}{3} - \frac{9}{4}\ r$$

Therefore the composite Corrector Integration Formula (2.3) becomes

$$y_{n+2} - y_n = h\ [(1-r)y'_n + y'_{n+2} + ry'_{n+3}]$$
$$+ h\ [(\frac{1}{3} - \frac{3}{4}\ r)y''_n - (\frac{1}{3} + \frac{9}{4}\ r)y''_{n+2}]$$

using the scalar test problem (1.8), we obtain

$$\frac{y_{n+2}}{y_n} = \frac{rq.(\bar{y}_{n+3}/y_n) + 1\ (1-r)q + (\frac{1}{3} - \frac{3}{4}\ r)q^2}{1 - q + (\frac{1}{3} - \frac{9}{4}\ r\ )q^2} \qquad (2.27)$$

which can also be written for r as

$$r = \frac{1 + q + \frac{1}{3}\ q^2 - e^{2q}(1 - q + \frac{1}{3}\ q^2)}{\frac{3}{4}\ q^2(3e^{2q} + 1) - q(e^{3q}-1)} \qquad (2.28)$$

To unite both the Predictor and the Corrector formulae, we proceed as before on steps A3 through A5 of the CIF. From previous derivation, it has been established that our set of integration formulas are consistent, that is

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \frac{\bar{y}_{n+2}}{y_n} = \bar{R}(q)$$

leading to

$$\frac{\bar{y}_{n+3}}{y_n} = \left(\frac{\bar{y}_{n+2}}{y_n}\right)^{\frac{3}{2}} = R^*(q)$$

Finally from step A5 of the Algorithm, (2.27) becomes

$$\frac{y_{n+2}}{y_n} = \frac{rqR^*(q) + 1 + (1-r)q + (\frac{1}{3} - \frac{3}{4} r)q^2}{1 - q + (\frac{1}{3} + \frac{9}{4} r)q^2} \tag{2.29}$$

Equation (2.29) is the Composite Integration Formula of order 4 for stiff problems which allow exponential fittings.

For stability conditions which R(q) must satisfy, we check by analytical processes for

$$R(q) < 1$$

And from (2.29) we have

$$\frac{rqR^*(q) + (1+r)q - 3rq^2}{1 - q + (\frac{1}{3} + \frac{9}{4} r)q^2} < 0$$

taking limit as $q \to -\infty$, we have

$$\frac{\lim\limits_{q\to-\infty} \frac{r}{q} R^*(q) - 3r}{\frac{1}{3} + \frac{9}{4} r} < 0$$

this gives $r > 0$ or $r < -\frac{4}{27}$

however, we need to check the conditions satisfied by a,

that is, $R(q) - 1 < 0$ as $q \to -\infty$

from (2.27), we have

$$a < \tfrac{2}{3} \quad \text{or} \quad a > 1$$

hence, when $r > 0$, then $\lim\limits_{q\to-\infty} \frac{r}{q}.R^*(q)$ required that $a > 1$

However, parameters a and r are bounded by certain limiting values for $q \in (-\infty, 0)$ Thus as $q \to 0$ and $q \to -\infty$ simultaneously, we have from (2.27)

$$\lim_{q \to 0} a = 1 \qquad \lim_{q \to -\infty} a = \frac{4}{3}$$

similarly,

$$\lim_{q \to 0} r = \frac{16}{135} \qquad \lim_{q \to -\infty} r = \frac{4}{9},$$

that is $a \varepsilon (1, \frac{4}{3})$ while $r \varepsilon (\frac{16}{135}, \frac{4}{9})$.

Hence, we established analytically for stiff system whose solutions are in the open left half plane,

that is $a > 1$ and $r > 0$

Furthermore, evaluating the values of a and r for some samples s, we obtain table 2.2. That is as q decreases, parameters a and r are monotone increasing, as given below:

TABLE 2.2
Values of parameters a and r for order 4 scheme

q	a	r
-1	1.0648	0.16829
-2	1.1204	0.21710
-3	1.16297	0.25855
-20	1.30088	0.40833
-50	1.3201	0.42977
-100	1.3267	0.43707

By Theorem 2.1, we conclude that our integration formula (2.29), which is a MLMM of order 4, is A-stable within the range of values specified for the choices of parameters a and r.

The stability polynomial used to determine the zero- and absolute-stabilities of the methods revealed that by equation (1.9), the predictor formula (2.25) gives

$$\xi = \frac{1 + (2-a)q + (\frac{4}{3} - a)q^2}{1 - aq - (\frac{2}{3} - a)q^2}$$

Evaluating for all values of $a \epsilon (1, \frac{4}{3})$ in the left-half plane, we obtain

$$|\xi| < 1 \quad \text{for all } q \epsilon (-\infty, 0)$$

Thus, in general for all $q \epsilon (-\infty, 0)$ the predictor scheme (2.25) is zero-stable.

Also the stability polynomial for the Predictor-Corrector formula following (1.9) gives

$$\xi^2 = \frac{rq^2R^*(q) + 1 + (1-r)q + (\frac{1}{3} - \frac{3}{4}r)q^2}{1 - q + (\frac{1}{3} - \frac{9}{4}r)q^2}$$

Testing for values of $q \epsilon (-\infty, 0)$ we established that

$$|\xi| < 1, \quad \text{for all } q$$

that is, the Predictor-Corrector formula fitted to stiff scalar problem (1.8) is absolutely stable.

This order 4 formula tested on various problems is seen to be more accurate than many known methods for the same problems. This shall be illustrated in chapter 4.

CHAPTER THREE

PADE EXPONENTIAL FORMULAS

3.1 INTRODUCTION

The derivative of stiff integration formulas of orders higher than order 4 will involve a one step ratio y_{n+1}/y_n which can be replaced either by an exponential function or by a one step scheme.

Usually the predictor formula involving this ratio is implicit. Hence to retain two step implicit scheme, the ratio y_{n+1}/y_n in the predictor formula is replaced not with an exponential function, but with an one step implicit method expressed as a 1-1 Pade approximation function. However, since the formula given in this thesis is designed for stiff problems for which exponential fitting is permitted, the exponential function will also be an appropriate substitution for the ratio y_{n+1}/y_n.

3.2 ORDER 5 INTEGRATION FORMULA

The order 5 formula introduces a one-step scheme at f_{n+1} term. This is chosen so that two-step procedure is maintained on the left hand side of equation (2.2). Hence, α_1 and γ_1 are set to zero in the simultaneous equations required to fix the fifth order integration formula.

Fitting order 4 predictor scheme using (2.4) for $c_j = 0$, $j = 0,1,2,3,4$ we obtain

$$\alpha_0 = -1, \quad \alpha_1 = 0, \quad \alpha_2 = 1$$

$$\beta_0 = a \quad \beta_1 = 2(1-a) \quad \beta_2 = a \quad \text{(free parameter)}$$

$$\gamma_0 = -\frac{1}{6}(1-2a), \quad \gamma_1 = 0, \quad \gamma_2 = \frac{1}{6}(1-2a)$$

Hence by (2.2) we obtain the predictor formula as

$$y_{n+2} - y_n = h\,[af_n + 2(1-a)f_{n+1} + af_{n+2}] + \frac{1}{6}h^2\,[(1-3a)g_{n+2} - (1-3a)g_n]$$

By (1.8) we obtain

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + aq - \frac{1}{6}(1-3a)q^2 + 2(1-a)q.(y_{n+1}/y_n)}{1 - aq - \frac{1}{6}(1-3a)q^2} \qquad (3.1)$$

The one-step y_{n+1}/y_n term is replaced by an exponentially fitted one-step implicit scheme. For this purpose, consider the most accurate one step implicit scheme, namely, the **Trapezoidal rule** given by

$$y_{n+1} - y_n = \tfrac{1}{2}h\,(f_n + f_{n+1})$$

using (1.8) we obtain a 1-1 Pade approximation

$$\frac{y_{n+1}}{y_n} = \frac{1 + \frac{1}{2}q}{1 - \frac{1}{2}q} \qquad (3.2)$$

substituting in (3.1), we have

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (\frac{3}{2} - a)q + (\frac{5}{6} - a)q^2 + \frac{1}{12}(1 - 3a)q^3}{1 - (\frac{1}{2} + a)q + (a - \frac{1}{6})q^2 + \frac{1}{12}(1 - 3a)q^3}.$$

$$= \bar{R}(q)$$

However, replacing the ratio y_{n+1}/y_n in (3.1) by e^q according to the procedure in the previous chapter, we have

$$\frac{y_{n+2}}{y_n} = \frac{1 + aq - \frac{1}{6}(1-3a)q^2 + 2(1-a)qe^q}{1 - aq - \frac{1}{6}(1-3a)q^2}$$

Furthermore, from (3.1), we have

$$a = \frac{-1 + \frac{1}{6}q^2 - 2qe^q + (1 - \frac{1}{6}q^2)e^{2q}}{q(1+\frac{1}{2}q) - 2qe^q + q(1-\frac{1}{2}q)e^{2q}} \qquad (3.3)$$

Similarly, to derive an order 5 corrector formula, set the following unknowns in composite formula (2.3) as

$\alpha_1 = \gamma_1 = 0$, $\beta_3 = r$ (free parameter)

$\alpha_2 = +1$, $\alpha_0 = -1$

On solving the system generated by $c_j = 0$, $j = 0,1,\ldots,5$ we obtain

$\beta_0 = \frac{7}{15} + \frac{7}{2}r$ $\beta_1 = \frac{16}{15} - 9r$

$\beta_2 = \frac{7}{15} + \frac{9}{2}r$ $\gamma_0 = \frac{1}{15} + \frac{3}{2}r$ $\gamma_2 = -\frac{1}{15} - 9r$

Hence the Corrector Integration Formula is given by

$$y_{n+2} - y_n = h\left[(\frac{7}{15} + \frac{7}{2}r)f_n + (\frac{16}{15} - 9r)f_{n+1} + (\frac{7}{15} + \frac{9}{2}r)f_{n+2} + rf_{n+3}\right] + h^2\left[(\frac{1}{15} + \frac{3}{2}r)g_n + (-\frac{1}{15} - \frac{9}{2}r)g_{n+2}\right]$$

And with exponential fittng, it becomes

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{7}{15} + \frac{7}{2}r)q + (\frac{1}{15} + \frac{3}{2}r)q^2 + (\frac{16}{15} - 9r)q.(\bar{y}_{n+1}/y_n) + rq(\bar{y}_{n+3}/y_n)}{1 - (\frac{7}{15} + \frac{7}{2}r)q + (\frac{1}{15} + \frac{9}{2}r)q^2} \qquad (3.4)$$

with localising assumption, we obtain r from (3.4) as

$$r = \frac{1 + \frac{7}{15}q + \frac{1}{15}q^2 + \frac{16}{15}qe^q - (1 - \frac{7}{15}q + \frac{1}{15}q^2)e^{2q}}{-\frac{7}{2}q - \frac{3}{2}q^2 + 9qe^q + \frac{9}{2}qe^{2q}(q-1) - qe^{3q}}$$

From previous derivation, we established that step A3 of the algorithm gives

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \frac{\bar{y}_{n+2}}{y_n} = \bar{R}(q)$$

which leads to

$$\frac{\bar{y}_{n+3}}{y_n} = [\bar{R}(q)]^{\frac{3}{2}} = R_2(q)$$

and

$$\frac{\bar{y}_{n+1}}{y_n} = [\bar{R}(q)]^{\frac{1}{2}} = R_1(q) \tag{3.6}$$

Thus instead of introducing another pade approximation, we simply fit the predictor formula, which has an 'in-built' pade approximation, into (3.5) by using (3.6).

Hence the Predictor-Corrector formula of order 5 is

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{7}{15} + \frac{7}{2}r)q + (\frac{1}{15} + \frac{3}{2}r)q^2 + (\frac{16}{15} - 9r)q.R_1(q) + rq.R_2(q)}{1 - (\frac{7}{15} + \frac{9}{2}r)q + (\frac{1}{15} + \frac{9}{2}r)q^2} \tag{3.7}$$

As it is done for lower order formulas, we obtain the range of values of a ~~and~~ r for which $q \in (-\infty, 0)$

Now, $\lim_{q \to -\infty} a = 0$

To obtain the limiting value of a as $q \to 0$, we apply L'Hospital rule at five stages involving tedious and careful differentiations to give

$$\lim_{q \to 0} a = \frac{19}{27}$$

that is for $q \in (-\infty\ 0)$, $a \in (0, \frac{19}{27})$

similarly, for $q \in (-\infty\ 0)$, $r \in (-\frac{2}{45}, 0)$

The values of a and r for some samples S in the range q are given in Table 3.1 below:

TABLE 3.1

q	a	r
-1	0.464189	-0.001883
-10	0.39168	-0.026082
-20	0.36481	-0.034591
-30	0.35476	-0.037751
-40	0.34956	-0.039381
-50	0.34639	-0.040373
-1000	0.33399	-0.044237

These results, as it was done in lower orders guarantee A-Stability of the order 5 scheme.

3.3 ORDER 6 INTEGRATION FORMULA

Following the procedure for order 5, the Predictor formula is obtained by solving six simultaneous equations corresponding to equation (2.4) to give

$$y_{n+2} - y_n = h\,[(\tfrac{14}{15} - a)f_n + \tfrac{16}{15} f_{n+1} + af_{n+2}] + h^2[(\tfrac{2}{9} - \tfrac{1}{3}a)g_n + \tfrac{4}{3}(\tfrac{7}{15} - a)g_{n+1} + (\tfrac{4}{45} - \tfrac{1}{3}a)g_{n+2}] \tag{3.8}$$

Similarly, choosing $\beta_3 = r$ as the free parameter, we obtain the Corrector formula by equation (2.4) as

$$y_{n+2} - y_n = h\,[(\tfrac{7}{15} - 10r)f_n + (\tfrac{16}{15} - 9r)f_{n+1} + (\tfrac{7}{15} + 18r)f_{n+2} + rf_{n+3}] + h^2[(\tfrac{1}{15} - 3r)g_n - 18rg_{n+1} - (\tfrac{1}{15}+9r)g_{n+2} \tag{3.9}$$

On fitting to the scalar problem (1.8), the Predictor formula (3.8) becomes

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (\frac{14}{15} - a)q + (\frac{2}{9} - \frac{1}{3}a)q^2 + [\frac{16}{15}q + (\frac{28}{45} - \frac{4}{3}a)q^2]\cdot\frac{y_{n+1}}{y_n}}{1 - aq - (\frac{4}{45} - \frac{1}{3}a)q^2} \tag{3.10}$$

using a 1-1 pade approximation (3.2), we obtain

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (\frac{3}{2} - a)q + (\frac{41}{45} - \frac{7}{6}a)q^2 + (\frac{1}{5} - \frac{1}{2}a)q^3}{1 - (\frac{1}{2} + a)q - (\frac{4}{45} - \frac{5}{6}a)q^2 + (\frac{2}{45} - \frac{1}{6}a)q^3}$$

$$= \bar{R}(q)$$

Replace the ratio y_{n+1}/y_n by e^q in (3.10) the sixth order Predictor formula can be expressed as

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{14}{15} - a)q + (\frac{2}{9} - \frac{1}{3}a)q^2 + [\frac{16}{15}q + (\frac{28}{45} - \frac{4}{3}a)q^2]e^q}{1 - aq - \frac{1}{3}(\frac{4}{45} - a)q^2}$$

obtaining parameter a from (3.10), we have

$$a = \frac{1 + \frac{14}{15}q + \frac{2}{3}q^2 + \frac{4}{15}qe^q(12+7q) + \frac{1}{15}e^{2q}(4q^2 - 45)}{q(3+q) + 4q^2e^q - qe^{2q}(3-q)} \qquad (3.11)$$

In a similar manner we fit (1.8) into (3.9) to get the corrector integration scheme as

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{7}{15} - 10r)q + (\frac{1}{15} - 3r)q^2 + [(\frac{16}{15} - 9r)q - 18rq^2].[\bar{R}(q)]^{\frac{1}{2}} + rq.[\bar{R}(q)]^{\frac{3}{2}}}{1 - (\frac{7}{15} + 18r)q + (\frac{1}{15} + 9r)q^2} \qquad (3.12)$$

and obtaining r from (3.9) we have

$$r = \frac{1 + \frac{7}{15}q + \frac{1}{15}q^2 + \frac{16}{15}qe^q - e^{2q}(1 - \frac{7}{15}q + \frac{1}{15}q^2)}{q(10+3q) + 9qe^q(1+2q) - 9qe^{2q}(2-q) - qe^{3q}} \qquad (3.13)$$

To obtain the range of values of a and r for which the order 6 Predictor -Corrector formula may be A-Stable, we apply L'Hospital rule for six derivatives on equation (3.11) to get

$$\lim_{q \to 0} a = \frac{7}{15}$$

and $$\lim_{q \to -\infty} a = \frac{2}{3}$$

Similarly, 7 times application of L'Hospital rule on (3.13) gives

$$\lim_{q \to 0} r = \frac{4}{945}$$

while $$\lim_{q \to -\infty} r = \frac{1}{45}$$

hence for $q \in (-\infty, 0)$ we have $a \in (\frac{7}{15}, \frac{2}{3})$ and $r \in (\frac{4}{945}, \frac{1}{45})$.

These show that as q increases, both a and r decreases. Following previous work, we established that within these ranges of values of a and r, the Predictor-Corrector formula (3.12) will be A-Stable for all choices of parameters a and r.

3.4 ORDER 7 SCHEME

Finally, the most accurate second derivative LMM to be given is order seven scheme. We hereby derive an order 7 Predictor-Corrector formula. The formula is derived without any free parameter as this is not required by the formula of this order.

However, obtaining a predictor formula of order 6 from (2.4), we obtain

$$y_{n+2} - y_n = \frac{h}{15}(7f_n + 16f_{n+1} + 7f_{n+2}) + \frac{h^2}{15}(g_n - g_{n+2})$$

Similarly, for order 7 Corrector scheme, eight equations for $c_j = 0$, $j = 0,1,\ldots,7$ obtained from (2.4) are solved to give

$$y_{n+2} - y_n = \frac{h}{35}\left(\frac{1049}{54} f_n + 34f_{n+1} + \frac{33}{2} f_{n+2} + \frac{2}{27} f_{n+3}\right) + \frac{h^2}{105}\left(\frac{37}{3} g_n + 4g_{n+1} - 8g_{n+2}\right) \qquad (3.14)$$

Equation (3.14) is the composite corrector formula.

Using exponential fitting, the predictor formula gives

$$\frac{\bar{y}_{n+2}}{y_n} \qquad \frac{15 + 7q + q^2 + 16q.(y_{n+1}/y_n)}{15 - 7q + q^2}$$

Using 1-1 pade approximation, we obtain

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{30 + 31q + 11q^2 - q^3}{30 - 29q + 9q^2 - q^3} \tag{3.15}$$

$$= \bar{R}(q)$$

substituting (3.15) into the corrector scheme (3.14) after using exponential fitting, we obtain the Predictor-Corrector formula of order 7 as

$$\frac{y_{n+2}}{y_n} = \frac{1 + \frac{1049}{1890} q + \frac{37}{315} q^2 - (\frac{34}{35} q + \frac{4}{105} q^2).[\bar{R}(q)]^{\frac{1}{2}} + \frac{2}{945} [\bar{R}(q)]^{\frac{3}{2}}}{1 - \frac{33}{70} q + \frac{8}{105} q^2}$$

These formulas derived so far in chapters 2 and 3 are coded in fortran language to solve several system of Ordinary Differential Equations, among which are the standard stiff Initial Value Problems discussed in the next chapter.

CHAPTER FOUR

COMPARATIVE ANALYSIS

4.1 INTRODUCTION

The formulas derived so far in the last two chapters are tested on several standard Stiff Initial Value Problems (IVPs) [2,6,7,8,11,16]. The aim of the computational analysis carried out in this chapter is firstly to show how these formulas compare fovourably with other known methods and the accuracy of these formulas when compared with the exact solution. Secondly, to demonstrate the performance of the step control procedure. For system of stiff IVPs, the eigenvalues are obtained from the Jacobian matrix of the system.

All computations given in this thesis are coded in a Fortran 77 on a 12/286MX, Leading Technology (IBM compatible) computer of Lagos State University, Ojo; using double precision arithmetic.

4.2 NUMERICAL EXAMPLES

PROBLEM 1

Consider the system of stiff IVP

$$\begin{aligned} y' &= -y + 95z, \quad & y(0) &= 1 \\ z' &= -y - 97z, \quad & z(0) &= 1 \end{aligned} \qquad x \in [0,1] \qquad (4.1)$$

Attempt will be made to demonstrate the efficiency of the formulas derived in chapters 2 and 3. Also given is the comparison of accuracy with other exponentially fitted schemes proposed by Kenue and Jackson [17] and Cash [2].

The eigenvalues of the Jacobian matrix of system (4.1) are

$\lambda_1 = -2$ and $\lambda_2 = -96$

and the general solution is of the form

$$y(x) = A e^{\lambda_1 x} + B e^{\lambda_2 x}$$
$$z(x) = C e^{\lambda_1 x} + D e^{\lambda_2 x} \qquad (4.2)$$

however, imposing another initial conditions at the derivatives of y and z, we obtain from (4.1)

$$y'(0) = 94$$
$$z'(0) = -98$$

and $e^{\lambda_2} \to 0$ as $x \to 0$

leaving us with $A = \frac{95}{47}$ and $c = \frac{-1}{47}$

Problem (4.1) is solved and tested on all our newly developed schemes of orders 2 to 7 using various step lengths.
For example, by using a step length $h = 0.0625$, we obtain $q = -0.125$ because $q = \lambda_1 h$.

The procedure for obtaining a result required the evaluation of parameters a and r and the quantity R(q). The program to solve this problem along with various output for orders two to seven are given in Appendix A.

Denote the second derivative two-step composite schemes derived in chapter 2 for orders 2 through 4 by SS2, SS3 and SS4 respectively. Furthermore, denote orders 5,6 and 7 formulas involving pade-approximations by PS5, PS6 and PS7, while those involving exact exponential fittings are denoted by ES5, ES6 and ES7 respectively.

Some results for the solution of problem 1 which are extracted from Appendix A are given in Table 4.1(a) below:

TABLE 4.1(a)

Accuracy table for MLMM on Problem 1

step size	method	max error y	max error z
0.03125	SS4	2.5×10^{-11}	2.6×10^{-13}
0.0625	SS2	5.6×10^{-17}	8.7×10^{-19}
	SS3	2.9×10^{-8}	3.1×10^{-10}
	SS4	4.4×10^{-16}	4.3×10^{-18}
	PS5	4.9×10^{-6}	5.2×10^{-8}
	ES5	5.6×10^{-17}	8.7×10^{-19}
	PS6	4.8×10^{-6}	4.8×10^{-8}
	PS7	6.0×10^{-3}	6.3×10^{-5}
	ES7	6.0×10^{-3}	6.3×10^{-5}
0.125	SS4	1.1×10^{-16}	8.7×10^{-19}
	ES5	1.1×10^{-16}	8.7×10^{-19}
0.25	SS2	0.0000	0.0000

The results above show that formulas SS2, SS4 and ES5 gave the least error for this problem. 1-1 Pade-approximation integrated into the formulas of orders 5, 6 and 7 affected the accuracy of their solutions, hence rather than using 1-1 Pade-approximation to estimate y_{n+1}/y_n, an exponential function e^q is rather preferred since it is exact for the exponential fittings. Hence ES5 gave high accuracy as orders 2 and 4 and identical to method SS4 for h = 0.125. Methods PS7 and ES7 performed poorly due to non existence of a free parameter which is an important factor in the derivation of other lower order schemes. Hence for a second derivative 2-step multistep methods, the maximal order 7 will perform poorly. However, orders 2 to 6 are sufficient to generate accurate solutions to any stiff problem for which exponential fitting is permitted.

We shall compare the results obtained by these methods with that of Cash [2] and Jackson and Kenue [17]. Let J-K denotes the method proposed by Jackson and Kenue, while SC4 and SC5 respectively denote the fourth and the fifth order schemes given by Cash for the case k = 1 of the MLMM. As shown in the Table 4.1(b) below, it will be observed that the methods proposed in this thesis are more accurate than the already existing ones.

TABLE 4.1(b)

Method	y(1)	z(1) x 10^{-2}	Error y	Error z
	h = 0.0625			
J-K	0.2725503	-0.2879477	$3x10^{-7}$	$4x10^{-9}$
SC4	0. 2735498	-0.2879471	$3x10^{-7}$	$3x10^{-9}$
SC5	0.27355005	-0.28794742	$1x10^{-8}$	$1x10^{-10}$
SS2	0.2735500406	-0.287947411	$6x10^{-17}$	$9x10^{-19}$
SS4	0.2735500406	-0.287947411	$3x10^{-16}$	$3x10^{-18}$
ES5	0.2735500405	-0.287947411	$6x10^{-17}$	$9x10^{-19}$
PS6	0.2735465656	-0.287943753	$4x10^{-6}$	$4x10^{-8}$
	h = 0.03125			
J-K	0.27355005	-0.28794742	$1x10^{-8}$	$1x10^{-10}$
SC4	0.27355003	-0.28794740	$1x10^{-8}$	$1x10^{-10}$
SS4	0.27355005	-0.287947402	$2x10^{-17}$	$2x10^{-19}$
True Sol.	0.2735500406	-0.287947411	---	---

PROBLEM 2 (Test problem given by Enright and Pryce [8])

Consider the 4 x 4 system of stiff Initial Value Problem

$$
\begin{aligned}
y_1' &= -10^4 y_1 + 100y_2 - 10y_3 + y_4 & y_1(0) &= 1 \\
y_2' &= -1000y_2 + 10y_3 - 10y_4 & y_2(0) &= 1 \\
y_3' &= -y_3 + 10y_4 & y_3(0) &= 1 \\
y_4' &= -0.1y_4 & y_4(0) &= 1
\end{aligned}
\tag{4.3}
$$

Putting in the matrix form, we obtain the eigenvalues of the Jacobian as

$$\lambda_1 = -0.1 \ , \quad \lambda_2 = -1 \ , \quad \lambda_3 = -1000 \ , \quad \lambda_4 = -10000$$

Following the arguement in Problem 1, the exact solution will be of the form

$$y_i(x) = v_i \, e^{\lambda_1 x} + w_i \, e^{\lambda_2 x} \ , \qquad i = 1,\ldots,4 \tag{4.4}$$

where

$$v_1 = \frac{-9909 - \lambda_2}{\lambda_1 - \lambda_2}$$

$$v_2 = \frac{-1000 - \lambda_2}{\lambda_1 - \lambda_2}$$

$$v_3 = \frac{-9 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$v_4 = \frac{-0.1 - \lambda_2}{\lambda_1 - \lambda_2}$$

$$w_1 = \frac{9909 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$w_2 = \frac{1000 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$w_3 = \frac{9 - \lambda_1}{\lambda_1 - \lambda_2}$$

$$w_4 = \frac{0.1 + \lambda_1}{\lambda_1 - \lambda_2}$$

The system (4.3) was considered by Enright and Pryce [8] and we adopted the error tolerance of 10^{-5}

The order 4 scheme which gives a very high accuracy in Problem 1 is used to solve the stiff system (4.3) over the interval $[0,1]$. Both versions of order 7 formula, that is PS7 and ES7, perform poorly as given in Appendix B. The error for y at x = 0.1 is poor with

$$y_1 = 20.0\ , \quad y_2 = 2.0\ , \quad y_3 = 0.04\ , \quad y_4 = 0.04$$

However, the result obtained using the order 4 formula derived in Chapter 2 is given in Table 4.2 below.

TABLE 4.2

Efficiency of Order 4 Scheme

Step Size	$y_1(1)$	$y_2(1)$	$y_3(1)$	$y_4(1)$
0.05	-5910.942866	-595.6555	6.334079	0.90483742
0.1	-5910.942866	-595.6555	6.334079	0.90483742
True Sol.	-5910.942866	-595.6555	6.334079	0.90483742
		ERROR		
0.05	4.5×10^{-12}	4.5×10^{-13}	3.6×10^{-15}	2×10^{-16}
0.1	3.6×10^{-12}	3.4×10^{-13}	3.1×10^{-15}	1×10^{-16}

The results obtained by order 4 show that the error tolerance can be raised to 10^{-12}.

The results obtained at x = 1 for h = 0.05 involved 10 steps while for h = 0.1 required only 5 steps. However, the results obtained for h = 0.1 are more accurate.

PROBLEM 3 (SECOND ORDER DIFFERENTIAL EQUATION)

Dalquist and Bjock [6] showed that there are some stiff problems for which Runge-Kutta (R-K) method is unsuitable. One of such problems is a second order differential equation

$$\begin{aligned} & y'' + 1001\, y' + 1000\, y = 0 \\ & y(0) = 1 \;, \quad y'(0) = -1 \end{aligned} \tag{4.5}$$

This problem has a general solution given by

$$y(x) = A\, e^{-x} + B\, e^{-1000x}$$

and for solution in [0,1] the exact solution is $y(x) = e^{-x}$. By setting $y' = z$, a 2 x 2 system of stiff IVP is obtained, that is

$$\begin{aligned} & y' = z \\ & z' = -1001\, z - 1000\, y \end{aligned} \tag{4.6}$$

with initial conditions $y(0) = 1$, $z(0) = -1$.

Explicit 4th order R-K method explodes for $h \Rightarrow 0.0027$. Although this is unsatisfactory step size for describing the function e^{-x} however on trying it on the new methods much improved result is obtained. The performance of this R-K method is due to its explicit nature which are generally unsuitable for stiff problems. The eigenvalues of the Jacobian matrix of (4.6) are $\lambda_1 = -1$, and $\lambda_2 = -1000$. This problem is solved (see Appendix C) using orders 3, 4 and 6 of the newly derived schemes and the results obtained at x = 1 are given in Table 4.3 below.

TABLE 4.3

Experimental Result on Second order ODE.

h	Formula	y(1)	Error
0.0027	R-K	0.367885	5.6×10^{-6}
	SS4	0.367879435	1.7×10^{-15}
0.05	SS3	0.367879435	2.0×10^{-10}
	SS4	0.367879435	1.7×10^{-16}
	PS6	0.367879436	5.6×10^{-8}
0.1	SS4	0.367879435	2.2×10^{-16}
	Exact Solution	0.367879435	----

It will be observed that for h = 0.0027 our new set of formulas including the less accurate order 6 method gave better accuracy than the explicit classical R-K formula of order 4. Other results obtained for this problem are given in Appendix C.

PROBLEM 4 (NON-LINEAR WITH REAL EIGENVALUES)

Chemical Kinetic Problem (Test problem from Enright and Pryce [8])

Gear [11] discussed the application of stiffly-stable integration formulae based on backward difference approximation of the derivative and used the following example to illustrate the method.

$$\begin{aligned} y_1' &= -0.013\, y_1 + 1000\, y_1 y_3 && y_1(0) = 1 \\ y_2' &= 2500\, y_2 y_3 && y_2(0) = 1 \qquad (4.7) \\ y_3' &= 0.013 y_1 - 1020 y_1 y_3 - 2500 y_2 y_3 && y_3(0) = 0 \end{aligned}$$

This application is from chemical kinetic reaction and y represents the concentration of a very reactive species which is an intermediate in the course of the reaction and always stays small.

y_1 and y_2 are monotonically decreasing and increasing respectively while y_3 increases to a maximum and thereafter is monotonically decreasing. Hull and Watt [16] showed that y_3 is bounded above by 1.3×10^{-5} while Enright and Pryce [8] suggested the error tolerance as 2.9×10^{-4}.

The eigenvalues of system (4.7) are given by

$$\lambda_1 = 0 \,, \quad \lambda_2 = -0.00928572 \text{ and } \lambda_3 = -3500.003714$$

The general solution of (4.7) is given by

$$y_i = A_i + B_i e^{\lambda_2 x} + C_i e^{\lambda_3 x} \qquad i = 1,2,3$$

for solution $x \in [0,1]$, as $x \to 1$, $e^{\lambda_3 x} \to 0$

Thus the exact solution is given by

$$y_i = A_i + B_i e^{\lambda_2 x}$$

where A_i and B_i are determined using the initial conditions.

The coded program and the set of solutions obtained for $x \in [0,1]$ are given in Appendix D. The error values at $x = 1$ when compared to the exact solution are given in Table 4.4 below.

TABLE 4.4

Experimental results on non-linear stiff problem

h	method	Error $y_1(1)$	Error $y_2(1)$	Error $y_3(1)$
0.0625	SS2	-4.4×10^{-16}	4.4×10^{-16}	1.3×10^{-15}
	SS4	4.4×10^{-16}	-4.4×10^{-16}	-8.9×10^{-16}
0.1	SS2	-6.7×10^{-16}	6.7×10^{-16}	1.8×10^{-15}
	SS4	-6.7×10^{-16}	6.7×10^{-16}	1.8×10^{-15}

It will be observed from Table 4.4 that for a step length $h = 0.1$, both schemes SS2 and SS4 are identical for y_1, y_2, and y_3 and very accurate when compared to the exact solution.

PROBLEM 5

A system of stiff problem which was considered both by Gear (see Hull et al [16]) and Enright [17] is discussed in this work. While these writers discussed function calls, Jacobian evaluations and matrix inversion for the efficiency of a method, the MLMM does not require any matrix inversion, instead a Predictor-Corrector formula of a given order is used. The Jacobian evaluation is only required for the determination of the eigenvalues.

Consider the system

$$\underline{y}' = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} \underline{y}, \quad \underline{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad (4.8)$$

$x \varepsilon [0,10]$

The error tolerance given by Enright is $\tau = 10^{-3}$

The eigenvalues are the non zero elements in the leading diagonal of the system, while the exact solution is

$$y_i(x) = A_i e^{\lambda_1 x} + B_i e^{\lambda_2 x}$$

where $\lambda_1 = -0.1$, $\lambda_2 = -10$ and the values of A_i and B_i determined from the initial conditions are given in Appendix E. For the purpose of comparison of results, Gear discussed variable-order, variable step stiffly-stable method, while Enright used a second derivative scheme. Denote the error of our second and fourth order formulas respectively by ER2 and ER4.

The tables of values below give solutions to problem (4.8) in the range $x \in [0,10]$

TABLE 4.5(a)

Efficiency of order 2 and 4 scheme

$h = 0.5$, $x \in [0,10]$

method	$y_1(10)$	$y_2(10)$	$y_3(10)$	$y_4(10)$
SS2	.3678794357	.37201x10	-3.344358	-36.7879435
SS4	.3678794357	.37201x10	-3.344358	-36.7879435
True Sol.	.3678794357	.37201x10	-3.344358	-36.7879435
ER2	3.3×10^{-16}	3.4×10^{-55}	3.1×10^{-15}	3.5×10^{-14}
ER4	1.7×10^{-16}	1.3×10^{-55}	1.3×10^{-15}	1.4×10^{-14}

TABLE 4.5(b)

Local error per unit step on various methods

method	steps	max local error/ unit step	Time/sec
GEAR	422	1.06×10^{-6}	0.72
ENRIGHT	125	0.44×10^{-6}	0.44
SS2	10	3.55×10^{-14}	0.50
SS4	10	1.42×10^{-14}	0.50

CHAPTER FIVE

CONCLUSION

5.1 SUMMARY

This research work shed some light into some manner by which accuracy can be improved for solution of stiff Initial Value Problems (IVPs). New algorithms have also been given to support this claim. In this thesis, solutions to stiff IVPs have been considered with methods based on an algorithm which allows exponential fitting. This suggests that the methods given in this thesis, basically are meant to solve stiff problems whose solutions can be expressed as exponential function.

In light of the work done, we observed that any of the schemes derived from orders 2 to 6 could be used for solutions of stiff IVPs. However, orders 2 and 4 seem to give higher accuracy than the others; though all the schemes still perform better than some of the existing methods for the same class of problems.

For order 3 formula, two parameters each were introduced as alternative approach to the predictor and the corrector schemes, thereby leading to two values for q. One of these is fixed while the other is varied for $q \in (-\infty, 0)$ This approach inroduced by Cash [2] may give good accuracy, but the work involved may not justify better accuracy over the illustrated approach given here. Hence, the two approaches were given in chapter 2, but a good choice of the free parameters a and r led to better accuracy which made our approach preferable.

5.2 SUGGESTION FOR FURTHER WORK

The MLMM discussed in this thesis permits us, not only to evaluate the second derivative of the dependent variable y, but may be extended to third or fourth derivative formula. The approach may eventually be extended to a general l-th order derivative. Furthermore, a new class of stiff problems having solutions in the form of trigonometric or hyperbolic function may be adapted and fitted into the sum of exponential functions. Since stiff problems have solutions with fast decaying exponents, then a solution on the left half plane may then be obtained within a small interval for which any of these oscillatory functions is decreasing.

The choice of free parameters may improve the accuracy of some methods both for 1-step MLMM and 2-step MLMM. Thus like in order 3 of our scheme, we observed that the free parameters are well chosen, so as to obtain such high accuracy. Hence, the positioning of the free parameters will affect the accuracy of the methods.

Finally, there is still a need to extend the scope of this work to derivation of methods which may be efficient when dealing with a wider class of stiff problems. However, we conclude and affirm that the methods given in this thesis is capable of solving all systems of stiff problems for which exponential fitting is applicable.

APPENDIX A

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 1

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

APPENDIX A

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(10),ZN(10), EXY(10),EXZ(10)
      OPEN(UNIT = 3, FILE='RES12', STATUS ='NEW')
      A1=95.0/17.0
      C1=-1.0/47.0
      D=-2.0
      HN=.25
      Q=D*HN
C TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C ISTEP METHOD FOR THE CASE K=2 ORDER 2 OF PROBLEM 1
      APN=(1.0-EXP(2.*Q))/Q+2.0
      APD=1.0-EXP(2.0*Q)
      A=APN/APD
      RPN=(EXP(2.*Q)-1.0)/Q-EXP(2.*Q)-1.0
      RPD=-1.5+EXP(2.*Q)+0.5*EXP(2.*Q)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q
      RQD=1.-A*Q
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,95)Q
      WRITE(3,10)A,R,RQ
      YN1=R+Q*(RQ+1.5)+1.0+0.5*Q*(2+R)
      YD=1.-0.5*Q*(2-R)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,2
      H=H+0.5
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.5
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PROBLEM 1 FOR ORDER 2 h=0.25',/)
   20 FORMAT(5X,'YN2= ',F18.12/)
   10 FORMAT(5X,'A R RQ '3(F18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   95 FORMAT(25X,' Q= 'F7.1,2X/)
  100 STOP
      END
```

RESULT OF PROBLEM 1 FOR ORDER 2   h=0.0625

Q= -.1250

A = 1.041623328376   B = -.182344792021   BQ = .778800783071

YB2= .778800783071E+00

```
H=   .125     Y=      .778800783071
YR=     1.574171795570      1.574171795570      .0000E+00
ZN=     -.016570229427      -.016570229427      .0000E+00
H=   .250     Y=      .606530659713
YR=     1.225966227079      1.225966227079      .0000E+00
ZN      .012904907653       .012904907653       .0000E+00
H    .375     Y=      .472366552741
YR=     .954783457668       .954783457668       .0000E+00
ZN=     .010050352186       .010050352186       .0000E+00
H=   .500     Y=      .367879441171
YR=     .743586104495       .743586104495       .0000E+00
ZN=     .007827222153       .007827222153       .0000E+00
H=   .625     Y=      .286504796860
YR=     .579105440462       .579105440462       .0000E+00
ZN=     .006095846712       .006095846712       .0000E+00
H=   .750     Y=      .223130160148
YR=     .451007770513       .451007770513       .0000E+00
ZN=     .004747450216       .004747450216       .0000E+00
H    .875     Y=      .173773943450
YR=     .351245204847       .351245204847       .5551E-16
ZN      .003697317946       .003697317946       .8674E-18
H   1.000     Y       .135335283237
YR=     .273550046585       .273550046585       .5551E-16
ZN      .002879471111       .002879471111       .8674E-18
```

RESULT OF PROBLEM 1 FOR ORDER 2   h=0.25

Q= .5000

X 1.16797113739   R = -.604971602672   RQ = .367879111171

YL .367879111171E+00

H .500   Y= .367879441171

YH .743586104195   .743586104195   .0000E+00

ZH .007827222153   .007827222153   .0000E+00

H= 1.000   Y= .135335283237

YH= .273550040585   .273550040585   .0000E+00

ZH= .002879471111   .002879171111   .0000E+00

```
      IMPLICIT real*8 (a-h,o-z)
      dimension yn(16),zn(16),exy(16),exz(16)
      open(3,file='evalte')
      a1=95.0/47.0
      c1=-1.0/47.0
      d=-2.0
      hh=0.03125
      q=d*hh
c     to evaluate y(x) and z(x) using multiderivative linear multis
c     method for the case k=2.
      arn=1.0+2.0*q+4.0/3.0*(q**2)+exp(2.0*q)*(2.0*(q**2)-3.0)/3.0
      ard=q**2*exp(2.0*q)-q*exp(2.0*q)+q*(q+1.0)
      a=arn/ard
      ren=1.0+q+(q**2)/3.0-exp(2.0*q)*(1.0-q+(q**2)/3.0)
      red=3.0*(q**2)*(3.0*exp(2.0*q)+1.0)/4.0-q*(exp(3.0*q)-1.0)
      r=ren/red
      ran=1.0+(2.0-a)*q+(4.0/3.0-a)*(q**2)
      rad=1.0-a*q-(2.0/3.0-a)*(q**2)
      rq=ran/rad
      write(3,85)arn,ard,ren,red
      write(3,10)a,r,rq
      yn1=r*q*(rq**1.5)+1.0+(1.0-r)*q+(1.0/3.0-3.0*r/4.0)*(q**2)
      yd=1.0-q+(1.0/3.0+9.0*r/4.0)*(q**2)
      yn2=yn1/yd
      write(3,20)yn2
      y=1.0
      z=1.0
      h=0.0
      x=0.0
      do 30 i=1,16
      h=h+0.0625
      y=y*yn2
      yn(i)=y*a1
      zn(i)=y*c1
      write(3,40)h,y
      x=x+0.0625
      exy(i)=a1*exp(d*x)
      ery=exy(i)-yn(i)
      exz(i)=c1*exp(d*x)
      erz=exz(i)-zn(i)
      write(3,65)yn(i),exy(i),ery
      write(3,70)zn(i),exz(i),erz
   30 continue
   20 format(5x,f18.12)
   10 format(5x,3(f18.12,2x)/)
   40 format(5x,'h= ',f6.4,5x,'y= ',f15.12,/)
   65 format(5x,'yn= ',f15.12,5x,f15.12,5x,e12.5,/)
   70 format(5x,'zn= ',e15.12,5x,f15.12,5x,e12.5,//)
   85 format(5x,4f15.10,2x/)
   80 stop
      end
```

THE RESULT FOLLOWING IS FOR ORDER 4 SCHEME FOR h=0.03125

```
        .0000095992          .0000095599          .0000000398          .0000003278
        1.004182438268       .121495731331        .882496902649
        .882496902598
   h= .0625        y=  .882496902588
   yn= 1.783770437982       1.783770437975       -.73475E-11
   zn= -.18776529247E-01    -.018776529248        .77341E-13

   h= .1250        y=  .779800783078
   yn= 1.574171886448       1.574171886435       -.12969E-10
   zn= -.16570228902SE-01   -.016570228902        .13651E-12

   h= .1875        y=  .687289278799
   yn= 1.389201813932       1.389201813914       -.17167E-10
   zn= -.146231756816E-01   -.014623175681        .18070E-12

   h= .2500        y=  .606530659723
   yn= 1.225966297865       1.225966297844       -.20200E-10
   zn= -.129049072450E-01   -.012904907245        .21263E-12

   h= .3125        y=  .535261428530
   yn= 1.081911460543       1.081911460521       -.22283E-10
   zn= -.113885406719E-01   -.011388540672        .23495E-12

   h= .3750        y=  .472366552753
   yn= .954783512804        .954783512780        -.23597E-10
   zn= -.100503519680E-01   -.010050351960        .24339E-12

   h= .4375        y=  .41686201[illegible]591
   yn= .8425[illegible]392   .8425934[illegible]660   -.24295E-10
   zn= -.886940439342E-02   -.008869404393        .25574E-12

   h= .5000        y=  .[illegible]678[illegible]41194
   yn= .7435[illegible]42   .7435[illegible]   .24503E-10
   zn= [illegible]          [illegible]           .25793E-12

   h= .5625        y=  .32465246737O
```

yn= .656212467925 .656212421900 .24327E-10

zn= -.690Z429087Z03E-02 -.006907499087 .25608E-12

h= .6250 y= .286504Z96872

yn= .579105473913 .579105473889 -.23854E-10

zn= -.609584654093E-02 -.006095846549 .25110E-12

h= .6875 y= .252839525816

yn= .511058797000 .511058796177 -.23156E-10

zn= -.537956569809E-02 -.005379565698 .24375E-12

h= .7500 y= .223130160152

yn= .451007796569 .451007796546 -.22293E-10

zn= -.474745006583E-02 -.004747450066 .23467E-12

h= .8125 y= .196911675215

yn= .398012983515 .398012983493 -.21313E-10

zn= -.418960997829E-02 -.004189609978 .22435E-12

h= .8750 y= .173773943460

yn= .351245225142 .351245225121 -.20256E-10

zn= -.369731702[illegible]E-02 -.003697317[illegible] .21322E-12

h= .9375 y= .153354966854

yn= .309972823236 .309972823217 -.19152E-10

zn= -.32620715[illegible]E-02 -.00326207[illegible] .20[illegible]0E-12

h= 1.0000 y= .[illegible]

yn= .2735500[illegible] .[illegible] -.18029E-10

zn= -.2879474020[illegible]E-02 -.00[illegible]02[illegible] .18[illegible]E-12

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES2', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.125
      Q=D*HH
C    TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C    ISTEP METHOD FOR THE CASE K=2 ORDER 4 OF PROBLEM 1
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      YN1=R*Q*(RQ**1.5)+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,4
      H=H+0.25
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.25
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PROBLEM 1 FOR ORDER 4 h=0.125',/)
   20 FORMAT(5X,'YN2= ',E18.12/)
   10 FORMAT(5X,'A R RQ '3(F18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(25X,' Q= 'F7.4,2X/)
  100 STOP
      END
```

RESULT OF PROBLEM 1 FOR ORDER 4 h=0.0625

Q= -.1250

A = 1.008329615677   R = .124500218663   RQ = .778800783071

YN2= .778800783071E+00

H= .125   Y= .778800783071

YN= .157417179557E+01   .157417179557E+01   .2220E-15

ZN= -.165702294271E-01   -.165702294271E-01   -.3469E-17

H= .250   Y= .606530659713

YN= .122596622708E+01   .122596622708E+01   .2220E-15

ZN= -.129049076535E-01   -.129049076535E-01   -.3469E-17

H= .375   Y= .472366552741

YN= .954783457668E+00   .954783457668E+00   .2220E-15

ZN= -.100503521860E-01   -.100503521860E-01   -.3469E-17

H= .500   Y= .367879441171

YN= .743586104495E+00   .743586104495E+00   .3331E-15

ZN= -.782722215258E-02   -.782722215258E-02   -.3469E-17

H= .625   Y= .286504796860

YN= .579105440462E+00   .579105440462E+00   .4441E-15

ZN= -.609584674171E-02   -.609584674171E-02   -.4337E-17

H= .750   Y= .223130160148

YN= .451007770513E+00   .451007770513E+00   .3331E-15

ZN= -.474745021592E-02   -.474745021592E-02   .3469E-17

H= .875   Y= .173773943450

YN= .351245204847E+00   .351245204847E+00   .3331E-15

ZN= -.369731794575E-02   -.369731794575E-02   -.3469E-17

H= 1.000   Y= .135335283237

YN= .273550040585E+00   .273550040585E+00   -.2776E-15

ZN= .287947411142E-02   .287947411142E-02   -.3036E-17

2
1

RESULT OF PROBLEM 1 FOR ORDER 4 h=0.125

```
                    Q=   -.2500

A = 1.016636987186  R = .130583414558     RQ = .606530659713

YN2=  .606530659713E+00

H=  .250     Y=        .606530659713

YN=       1.225966227079      1.225966227079       .0000E+00

ZN=       -.012904907653      -.012904907653       .0000E+00

H=  .500     Y=        .367879441171

YN=        .743586104495       .743586104495       .0000E+00

ZN=       -.007827222153      -.007827222153       .0000E+00

H=  .750     Y=        .223130160148

YN=        .451007770513       .451007770513      -.1110E-15

ZN=       -.004747450216      -.004747450216       .8674E-18

H= 1.000     Y=        .135335283237

YN=        .273550040585       .273550040585       .0000E+00

ZN=       -.002879474111      -.002879474111       .0000E+00
```

```fortran
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES5', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.0625
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 5 ON PROBLEM 1
      APN=-1.+(1./6.)*(Q**2)-2.*Q*EXP(Q)+(1.-(Q**2)/6.)*EXP(2.*Q
     1)
      APD=Q*(1.+0.5*Q)-2.*Q*EXP(Q)+Q*(1.0-0.5*Q)*EXP(2.*Q)
      A=APN/APD
      RPN=1.0+(7./15.)*Q+(Q**2)/15.+16.*Q/15.*EXP(Q)-(1.-7.*Q/1
     15.+(Q**2)/15.)*EXP(2.*Q)
      RPD=-0.5*Q*(7.+3.*Q)+9.*Q*EXP(Q)-4.5*Q*EXP(2.*Q)*(1.-Q)-Q*
     1EXP(3.0*Q)
      R=RPN/RPD
      RQN=1.+(1.5-A)*Q+(5./6.-A)*(Q**2)+(1./12.-0.25*A)*(Q**3)
      RQD=1.-(0.5+A)*Q+(A-1./6.)*(Q**2)+(1./12.-0.25*A)*(Q**3)
C     EVALUATE R(q) USING PADE APPROX.FOR Y(N+1)/Y(N)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      R1Q=RQ**0.5
      R2Q=RQ**1.5
      YN1=1.+Q*(7./15.+3.5*R)+(1./15.+1.5*R)*(Q**2)+(16./15.-9.*R
     1)*Q*R1Q+R*Q*R2Q
      YD=1.-(7./15.+4.5*R)*Q+(1./15.+4.5*R)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,8
      H=H+0.125
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.125
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'PADE-RESULT OF PROBL 1 FOR ORDER 5 H=0.0625'
   20 FORMAT(5X,'YN2= ',E18.12/)
   10 FORMAT(5X,'A R RQ '3(E18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN=',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(25X,'Q = 'F7.4,2X/)
  100 STOP
      END
```

```
EXPO. RES PRB 1. CASE K=2 ORDER 5 h=0.0625

                    Q=  -.1250

A = .466626999630    R = -.000183739779    RQ = .778800783071

YN2=  .778800783071E+00

H=  .125      Y=        .778800783071

YN=     1.574171795570         1.574171795570

ZN=        .016570229427        -.016570229427        .0000E+00

H=  .250      Y=        .606530659713                 .0000E+00

YN=     1.225966227079         1.225966227079

ZN=       -.012904907653        -.012904907653        .0000E+00

H=  .375      Y=        .472366552741                 .0000E+00

YN=      .954783457668          .954783457668

ZN=      -.010050352186         -.010050352186        .0000E+00

H=  .500      Y=        .367879441171                 .0000E+00

YN=      .743586104495          .743586104495

ZN=      -.007827222153         -.007827222153        .0000E+00

H=  .625      Y=        .286504796860                 .0000E+00

YN=      .579105440462          .579105440462

ZN=       .006095846742         -.006095846742        .0000E+00

H=  .750      Y=        .223130160148                 .0000E+00

YN=      .451007770513          .451007770513

ZN=       .004717150216          .004747450216        .0000E+00

H=  .875   ,  Y=        .173773943450                 .0000E+00

YN=      .351245204847          .351245204847

ZN=      -.003697317946         -.003697317946        .5551E-16

H= 1.000      Y=        .135335283237                 .8674E-18

YN=      .273550040585          .273550040585

ZN=      -.002879474111         -.002879474111        .5551E-16

                                                      -.8674E-18
```

PADE-RESULT OF PROBL 1 FOR ORDER 5 h=0.0625

Q = -.1250

A = .466626999630E+00 R= -.183739778859E-03 RQ=.778818903546E+00

YN2= .778799489296E+00

H= .125	Y= .778799489296		
YN=	1.574169180493	1.574171795570	.2615E-05
ZN=	-.016570201900	-.016570229427	-.2753E-07
H= .250	Y= .606528644528		
YN=	1.225962153834	1.225966227079	.4073E-05
ZN=	-.012904864777	-.012904907653	-.4288E-07
H= .375	Y= .472364198602		
YN=	.954778699303	.954783457668	.4758E-05
ZN=	-.010050302098	-.010050352186	-.5009E-07
H= .500	Y= .367876996633		
YN=	.743581163408	.743586104495	.4941E-05
ZN=	-.007827170141	-.007827222153	-.5201E-07
H= .625	Y= .286502417102		
YN=	.579100630313	.579105440462	.4810E-05
ZN=	-.006095796109	-.006095846742	-.5063E-07
H= .750	Y= .223127936121		
YN=	.451003275139	.451007770513	.4495E-05
ZN=	.004747402896	-.004747450216	-.4732E-07
H= .875	Y= .173771922699		
YN=	.351241120349	.351245204847	.4084E-05
ZN=	-.003697274951	-.003697317946	-.4299E-07
H= 1.000	Y= .135333484652		
YN=	.273546405148	.273550040585	.3635E-05
ZN=	.002879435844	-.002879474111	-.3827E-07

EXPO. RES PRB 1, CASE K=2 ORDER 5 h=0.125

Q= -.2500

A = .466508184129 R = -.000382298798 RQ = .606530659713

YN2= .606530659713E+00

H= .250	Y=	.606530659713	
YN=	1.225966227079	1.225966227079	.0000E+00
ZN=	-.012904907653	-.012904907653	.0000E+00
H= .500	Y=	.367879441171	
YN=	.743586104495	.743586104495	.0000E+00
ZN=	-.007827222153	-.007827222153	.0000E+00
H= .750	Y=	.223130160148	
YN=	.451007770513	.451007770513	-.1110E-15
ZN=	-.004747450216	-.004747450216	.8674E-18
H= 1.000	Y=	.135335283237	
YN=	.273550040585	.273550040585	.0000E+00
ZN=	-.002879474111	-.002879474111	.0000E+00

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='EVALTE6', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=0.05
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 6
      APN=3.0+14.0*Q/5.+(2./3.)*(Q**2)+(4.*Q/15.)*EXP(Q)*(12.0+7.
     10*Q)+EXP(2.0*Q)*(4.*(Q**2)-45.0)/15.0
      APD=Q*(3.0*Q)+4.*(Q**2)*EXP(Q)-Q*EXP(2.*Q)*(3.-Q)
      A=APN/APD
      RPN=1.+7.*Q/15.+(Q**2)/15.+16.*Q*EXP(Q)/15.-EXP(2.*Q)*(1.0-
     17.*Q/15.+(Q**2)/15.0)
      RPD=Q*(10.+3.*Q)+9.*Q*EXP(Q)*(1.+2.*Q)-9*Q*EXP(2.*Q)*(2.-Q)
     1-Q*EXP(3.0*Q)
      R=RPN/RPD
      RQN=1.0+Q*(1.5-A)+(Q**2)*(41./45.-7.*A/6.)+(Q**3)*(.2-.5*A)
      RQD=1.0-Q*(A+0.5)-(Q**2)*(4.0/45.-5.0*A/6.0)+(Q**3)*(2./45
     1.0-A/6.0)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      S=RQ**0.5
      T=RQ**1.5
      YN1=1.0+(7./15.-10.0*R)*Q+(1./15.-3.*R)*(Q**2)+R*Q*T+S*((1
     16./15.-9.*R)*Q-18.*R*(Q**2))
      YD=1.-(7/15.+18.*R)*Q+(1./15.+9.*R)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.1
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.1
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
30    CONTINUE
99    FORMAT(5X,'RESULT OF PRBLM(I) FOR CASE K=2 ORDER 6 h=.05',/)
20    FORMAT(5X,'YN2= 'E18.12)
10    FORMAT(5X,'A R RQ  ',3(E12.6,2X)/)
40    FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
65    FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
70    FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
85    FORMAT(25X,'Q = ',F6.4,2X/)
100   STOP
      END
```

```
PADE-RESULT OF PROBLEM 1 FOR ORDER 6  h=.0625

                Q = -.1250

A R RQ    .129658E-06    .438503E-02    .778819E+00

YN2=  .778799546412E+00

H=  .125      Y=        .778799546412

YN=       1.574169295939      1.574171795570      .2500E-05

ZN=       -.016570203115      -.016570229427     -.2631E-07

H=  .250      Y=        .606528733492

YN=       1.225962333653      1.225966227079      .3893E-05

ZN=       -.012904866670      -.012904907653     -.4098E-07

H=  .375      Y=        .472364302529

YN=        .954778909367       .954783457668      .4548E-05

ZN=       -.010050304309      -.010050352186     -.4788E-07

H=  .500      Y=        .367877104551

YN=        .743581381539       .743586104495      .4723E-05

ZN=       -.007827172437      -.007827222153     -.4972E-07

H=  .625      Y=        .286502522160

YN=        .579100842663       .579105440462      .4598E-05

ZN=       -.006095798344      -.006095846742     -.4840E-07

H=  .750      Y=        .223128034304

YN=        .451003473593       .451007770513      .4297E-05

ZN=       -.004747404985      -.004747450216     -.4523E-07

H=  .875      Y=        .173772011908

YN=        .351241300664       .351245204847      .3904E-05

ZN=       -.003697276849      -.003697317946     -.4110E-07

H= 1.000      Y=        .135333564053

YN=        .273546565639       .273550040585      .3475E-05

ZN=       -.002879437533      -.002879474111     -.3658E-07
```

PADE-RESULT OF PROBLEM 1 FOR ORDER 6 h=.05

Q = -.1000

A R RQ .441857E-07 .435442E-02 .818738E+00

YN2= .818730334674E+00

H= .100	Y= .818730334674		
YN=	1.654880463704	1.654881304481	.8408E-06
ZN=	-.017419794355	-.017419803205	-.8850E-08
H= .200	Y= .670319360916		
YN=	1.354900835894	1.354902212634	.1377E-05
ZN=	-.014262114062	-.014262128554	-.1449E-07
H= .300	Y= .548810794702		
YN=	1.109298414823	1.109300105591	.1691E-05
ZN=	-.011676825419	-.011676843217	-.1780E-07
H= .400	Y= .449328045619		
YN=	.908216262422	.908218108134	.1846E-05
ZN=	-.009560171183	-.009560190612	-.1943E-07
H= .500	Y= .367878501168		
YN=	.743584204489	.743586093415	.1889E-05
ZN=	-.007827202153	-.007827222036	-.1988E-07
H= .600	Y= .301193288381		
YN=	.608794944600	.608796800426	.1856E-05
ZN=	-.006408367838	-.006408387373	-.1954E-07
H= .700	Y= .246596081798		
YN=	.498438888741	.498440661399	.1773E-05
ZN=	-.005246725145	-.005246743804	-.1866E-07
H= .800	Y= .201895692580		
YN=	.408087038193	.408088696855	.1659E-05
ZN=	-.004295653034	-.004295670493	-.1746E-07
H= .900	Y= .165298127955		
YN=	.334113237356	.334114765103	.1528E-05
ZN=	-.003516981446	-.003516997527	-.1608E-07
H= 1.000	Y= .135334591622		
YN=	.273548642640	.273550032432	.1390E-05

ZN= -.002879459396 -.002879474026 -.1463E-07

```
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
OPEN(UNIT = 3, FILE='RES8',STATUS = 'NEW')
A1=95.0/47.0
C1=-1.0/47.0
D=-2.0
HH=.0625
Q=D*HH
C EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C STEP METHOD FOR THE CASE K=2 ORDER 7 FOR PROBLEM 1
C REPLACE Y(n+1)/Y(n) BY EXP Q
RQN=15.+7.*Q+Q**2+16.*Q*EXP(Q)
RQD=15.0-7.*Q+Q**2
RQ=RQN/RQD
WRITE(3,99)
WRITE(3,10)Q,RQ
R1Q=RQ**0.5
R2Q=RQ**1.5
YN1=1.0+1049.*Q/1890.+37.*(Q**2)/315.+2.*R2Q/945.+R1Q*(34.*
&Q/35.+4.*(Q**2)/105.)
YD=1.-33.*Q/70.+8*(Q**2)/105.
YN2=YN1/YD
WRITE(3,20)YN2
Y=1.0
Z=1.0
H=0.0
X=0.0
DO 30 I=1,8
H=H+0.125
Y=Y*YN2
YN(I)=Y*A1
ZN(I)=Y*C1
WRITE(3,40)H,Y
X=X+0.125
EXY(I)=A1*EXP(D*X)
ERY=EXY(I)-YN(I)
EXZ(I)=C1*EXP(D*X)
ERZ=EXZ(I)-ZN(I)
WRITE(3,65)YN(I),EXY(I),ERY
WRITE(3,70)ZN(I),EXZ(I),ERZ
CONTINUE
FORMAT(5X,'EXP FN RESULT FOR ORDER 7 h=.0625 OF PROBLEM 1',
1/)
FORMAT(5X,' YN2 =',E18.12/)
FORMAT(5X,' Q=',F10.4,5X,'RQ= 'F18.12,/)
FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
FORMAT(5X,4F15.10,2X/)
STOP
END
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES7',STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.005
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 7 ON PROBLEM 1
      RQN=30.+31.*Q+11.*(Q**2)-Q**3
      RQD=30.0-29.*Q+9.*(Q**2)-Q**3
C     EVALUATE R(q) USING PADE APPROXIMATION FOR Y(n+1)/Y(n)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,10)Q,RQ
      R1Q=RQ**0.5
      R2Q=RQ**1.5
      YN1=1.0+1049.*Q/1890.+37.*(Q**2)/315.+2.*R2Q/945.+R1Q*(34.*
     1Q/35.+4.*(Q**2)/105.)
      YD=1.-33.*Q/70.+8*(Q**2)/105.
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,20
      H=H+0.01
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.02
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'PADE APPROX.RESULT FOR ORDER 7 h=0.005 OF
     1PROBLEM 1',/)
   20 FORMAT(5X,'YN2= ',F18.12)
   10 FORMAT(5X,'Q=',F10.4,5X,'RQ=',F18.12,/)
   40 FORMAT(5X,'H=',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN=',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN=',F18.12,5X,F15.12,5X,E10.4,/)
  100 STOP
      END
```

EXP FN RESULT FOR ORDER 7 h=0.0625 OF PROBLEM 1

```
 Q=      -.1250        RQ=         .778800783156

 YN2 = .780361932486E+00

H=  .125        Y=        .780361932486

YN=       1.577327310344        1.574171795570       -.3156E-02

ZN=       -.016603445372        -.016570229427        .3322E-04

H=  .250        Y=        .608964745673

YN=       1.230886188063        1.225966227079       -.4920E-02

ZN=       -.012956696716        -.012904907653        .5179E-04

H=  .375        Y=        .475212905749

YN=        .960536724387         .954783457668       -.5753E-02

ZN=       -.010110912888        -.010050352186        .6056E-04

H=  .500        Y=        .370838061473

YN=        .749566294466         .743586104495       -.5980E-02

ZN=       -.007890171521        -.007827222153        .6295E-04

H=  .625        Y=        .289387906290

YN=        .584933002076         .579105440462       -.5828E-02

ZN=       -.006157189496        -.006095846742        .6134E-04

H=  .750        Y=        .225827305791

YN=        .456459447875         .451007770513       -.5452E-02

ZN=       -.004804836293        -.004747450216        .5739E-04

H=  .875        Y=        .176227032755

YN=        .356203576845         .351245204847       -.4958E-02

ZN=       -.003749511335        -.003697317946        .5219E-04

H= 1.000        Y=        .137520867837

YN=        .277967711585         .273550040585       -.4418E-02

ZN=       -.002925975911        -.002879474111        .4650E-04
```

```
PADE APPROX.RESULT FOR ORDER 7 h=0.0625 OF PROBLEM 1

[illegible]      .125000      .778818902192

YN2          .780360810273

H     .125        Y=         .780360810273

YN=       1.577325042042        1.574171795570        .3153E 02

ZN=        .016603121195         .016570229427        .3319E 04

H     .250        Y=         .608962994210

YN=       1.230032617872        1.225966227079        .4916E 02

ZN=        .012956659151         .012904907658        .5155E 01

H     .375        Y=         .475210855580

YN=        .960532580415         .951783457608        .5749E 02

ZN=        .010110869268         .010050352186        .6052E 01

H=    .500        Y=         .370835928318

YN=        .719561982770         .743586161195        .5976E 02

ZN=        .007890126134         .007827222153        .6290E 04

H=    .625        Y=         .288305825500

YN=        .584928796224         .579105440162        .5823E 02

ZN=        .006157145223        -.006095846742        .6130E 01

H=    .750        Y=         .225825357269

YN=        .456455509374         .451007770513        .5148E 02

ZN=        .004801791336        -.004747450216        .5734E 04

H     .875        Y=         .176225258779

YN=        .356199991149         .351245204847        .1955E 02

ZN=        .003719173591         .003697317946        .5216E 04

H=   1.000        Y=         .137519285731

YN=        .277961513712         .273550010565        .4414E 02

ZN=        .002925942250         .002879471111        .4647E 04
```

```
EXP FN RESULT FOR ORDER 7 h=0.005 OF PROBLEM 1

Q=      .0100       RQ=         .980198673715

YN2 = .982263325691E+00

H=   .010       Y=         .982263325691

YN=       1.985125871077        1.981252638421        .4173E-02

ZN=        .020899219696        -.020855290931        .4393E-04

H=   .020       Y=         .964841240997

YN=       1.950211019037        1.942021203534        .8190E-02

ZN=        .020528537042        -.020142328511        .8621E-04

H=   .030       Y=         .917728166146

YN=       1.915620761359        1.903566612989        .1205E-01

ZN=        .020164429067         .020037513295        .1269E-03

H=   .040       Y=         .930918620329

YN=       1.881614019815        1.865873169437        .1577E-01

ZN=        .019806779156         .019640773362        .1660E-03

H=   .050       Y=         .914407219952

YN=       1.848269912669        1.828926700118        .1931E-01

ZN=        .019455172765         .019251860001        .2036E-03

H=   .060       Y=         .898138676906

YN=       1.815187751193        1.792711525832        .2276E-01

ZN=        .019110397381         .018870647640        .2397E-03

H=   .070       Y=         .882257796876

YN=       1.783287036238        1.757213460023        .2607E-01

ZN=        .018771442487        -.018196903790        .2745E-03

H=   .080       Y=         .866609477676

YN=       1.751657454877        1.722418303006        .2924E-01

ZN=        .018138490525        -.018130718979        .3078E-03
```

H= .090	Y=	.851238707617	
YN=	1.720388877098	1.683312130211	.3228E 01
ZN=	.018111161364	.017771706697	.3398E 03
H= .100	Y=	.836140563901	
YN=	1.690071352566	1.654881316811	.3519E 01
ZN=	.017790221764	-.017419803335	.3704E 03
H= .110	Y=	.821310211042	
YN=	1.660035107426	1.622112471943	.3798E 01
ZN=	.017171685311	.017074868126	.3998E 03
H= .120	Y=	.806742899322	
YN=	1.630650511183	1.589992493664	.4066E 01
ZN=	.017161742539	.016736763091	.4280E 03
H= .130	Y	.792133963266	
YN=	1.601728223623	1.558508533551	.4322E 01
ZN=	.016860297091	.016405352985	.4549E 03
H= .140	Y	.778378820118	
YN=	1.573318891788	1.527647997610	.4567E 01
ZN=	.016561251493	.016080505233	.4807E 03
H .150	Y	.761572960526	
YN=	1.545412117020	1.497398541266	.4801E 01
ZN=	.016267502969	.015762089907	.5051E 03
H= .160	Y	.751011986798	
YN=	1.[illegible]02952030	1.467748061158	.5025E 01
ZN=	.01[illegible]78978413	.015119979623	.5290E 03
H= .170	Y	.737691531783	
YN=	1.[illegible]	1.[illegible]	.5239E 01
ZN=	.01[illegible]05561506	.0151140195[illegible]	.5515E 03

H= .180 Y= .724607337346

YN= -.010973489411 -.010565644927 .4078E-03

ZN= -.015417177390 -.014844177270 .5730E-03

H= .190 Y= .711755213001

YN= -.010778856203 -.010356431145 .4224E-03

ZN= -.010565420555 -.010151360073 .4141E-03

H= .200 Y= .699131042600

YN= -.010587675141 -.015113334187 -.4526E-02

ZN= -.010378025131 -.014814070126 -.4436E-02

H= .210 Y= .686730782998

YN= -.010399884995 -.015111800056 -.4712E-02

ZN= -.010193953480 -.014812566373 -.4619E-02

H= .220 Y= .674550462762

YN= -.010215425622 -.015110266081 -.4895E-02

ZN= -.010013146647 -.014811062772 -.4798E-02

H= .230 Y= .662586180899

YN= -.010034237945 -.015108732261 -.5074E-02

ZN= -.009835546726 -.014809559324 -.4974E-02

H= .240 Y= .650834105607

YN= -.009856263935 -.015107198597 -.5251E-02

ZN= -.009661096837 -.014808056029 -.5147E-02

H= .250 Y= .639290473047

YN= -.009681446591 -.015105665089 -.5424E-02

ZN= -.009489741109 -.014806552886 -.5317E-02

PADE APPROX.RESULT FOR ORDER 7 h=0.005 OF PROBLEM 1

Q= .0100 EQ= .980198674621

YN2= .982203325689

H= .010 Y= .982203325689

YN= 1.985125871074 1.942021208534 .4340E-01

ZN= .020899219696 -.020442328511 .4569E-03

H= .020 Y= .964841240994

YN= 1.950211019031 1.865873469437 -.8434E-01

ZN= .020528537042 -.019640773362 .8878E-03

H= .030 Y= .917728166141

YN= 1.915620761350 1.792711525832 .1229E100

ZN= .020161120067 .018870647610 .1294E 02

H= .040 Y= .930913620324

YN= 1.884611019803 1.722118303066 .1592E100

ZN= .019806779156 .018130718979 .1670E 02

H= .050 Y= .911107219945

YN= 1.848269912655 1.654881316311 .1934E100

ZN= .019175172765 .017419803335 .2036E 02

H= .060 Y= .893188676898

YN= 1.815187751176 1.589992493664 .2255E100

ZN= .019116157331 .016736703091 .2371E 02

H= .070 Y= .882257790366

YN= 1.783287036219 1.527647997610 .2556E100

ZN= .018771112437 .016080505238 .2691E 02

H= .080 Y= .866609177665

YN= 1.751657454855 1.467748064158 .2339E100

ZN= .018436499525 .015449979623 .2989E 02

```
H= .090      Y=      .851238707605

YN       1.726588877071       1.410196840640       .3101E+00

ZN=       .018111161361        .014814177270       .3267E-02

H= .100      Y=      .836140563888

YN=      1.690071352539       1.354902232824       .3352E+00

ZN=       .017790224764        .011262128767       .3528E-02

H= .110      Y=      .821310211028

YN=      1.660095107397       1.301775757545      -.3583E+00

ZN=       .017471665341       -.013702902711       .3772E-02

H= .120      Y=      .806742899307

YN=      1.630650511153       1.250732401112       .3799E+00

ZN=       .017164712538       -.013165604222       .3999E-02

H= .130      Y=      .792433963250

YN=      1.601728823590       1.201690483260       .4000E+00

ZN=       .016860257090       -.012649372508       .4211E-02

H= .140      Y=      .778370820131

YN=      1.573318891754       1.154571526487       .4187E+00

ZN=       .016561251492       -.012153384189       .4408E-02

H= .150      Y=      .764572968508

YN=      1.545113146984       1.109300130366       .4361E+00

ZN=       .016267509968        .011676843178       .4591E-02

H= .160      Y=      .751011986779

YN       1.[illegible]2552000       1.065803651070       .4522E+00

ZN=       .015976973412        .011218987966       .4760E-02

H= .170      Y=      .737691531766

YN=      1.491078628033       1.021013085239       .4671E+00

ZN=       .0156[illegible]61506       -.010779085108       .4916E-02
```

APPENDIX B

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 2

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SL3', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-1.0
      P=D(1)-D(2)
      V(1)=(-9909.0-D(2))/P
      V(2)=(-1000.0-D(2))/P
      V(3)=(9.0-D(2))/P
      V(4)=(-0.1-D(2))/P
      W(1)=(9909.0+D(1))/P
      W(2)=(1000.0+D(1))/P
      W(3)=(-9.0+D(1))/P
      W(4)=(0.1+D(1))/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.1
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 ON PROBLEM 2
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,11)Q
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,5
      H=H+0.2
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+0.2
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
```

```
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
      WRITE(3,80)Y3N(I),EXY3(I),ER3
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 4 h=0.1 ON PROBLEM 2',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'A R RQ',3(E15.6,2X)/)
   11 FORMAT(5X,' Q = ',2(F10.4,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'Y1= ',F18.12,5X,F18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2= ',F18.12,5X,F18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3= ',F18.12,5X,F18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4= ',F18.12,5X,F18.12,5X,E10.4,/)
  100 STOP
      END
```

```
RESULT FOR ORDER 4 h=0.05 ON PROBLEM 2

V(I)=  -.1101E+05     -.1111E+04      .1000E+02     -.1111E+00

W(I)=   .1101E+05      .1111E+04     -.1000E+02      .1111E+00

A R RQ     .100033E+01      .118728E+00      .990050E+00

A R RQ     .100333E+01      .120898E+00      .904837E+00

YP(I)=      .990049833454      .904837416688

H=  .100      Y=      .990049833454

Y1=  -938.188710152108      -938.188710152108      .0000E+00

Y2=   -94.680463230609       -94.680463230609      .0000E+00

Y3=      .852124169075          .852124169075      .0000E+00

Y4=     -.009468046464         -.009468046464      .0000E+00

H=  .200      Y=      .980198672723

Y1= -1777.761825094169     -1777.761825094169      .0000E+00

Y2=  -179.408802613197      -179.408802613197      .0000E+00

Y3=     1.614679223519         1.614679223519      .0000E+00

Y4=     -.017940880529         -.017940880529      .0000E+00

H=  .300      Y=      .970445532681

Y1= -2528.196745758465     -2528.196745758464      .1364E-11

Y2=  -255.141461878945      -255.141461878945      .1137E-12

Y3=     2.296273156911         2.296273156911     -.1332E-14

Y4=     -.025514146568         -.025514146568      .1388E-16

H=  .400      Y=      .960789438007

Y1= -3198.068054888938     -3198.068054888937      .1364E-11

Y2=  -322.743773830754      -322.743773830754      .1137E-12

Y3=     2.904693964477         2.904693964477     -.1332E-14

Y4=     -.032274377864         -.032274377864      .1388E-16

H=  .500      Y=      .951229423083

Y1= -3795.133440748525     -3795.133440748522      .2728E-11

Y2=  -382.998631622618      -382.998631622618      .2274E-12

Y3=     3.446987684604         3.446987684604     -.2220E-14

Y4=     -.038299863733         -.038299863733      .2776E-16

H=  .600      Y=      .941764531900
```



```
WRITE(6,70)YZN(I),EXYZ(I),ERZ
```

Y1=	-4326.411444013024	-4326.411444013021	.3638E-11
Y2=	-436.614334848423	-436.614334848423	.3979E-12
Y3=	3.929529013636	3.929529013636	-.3109E-14
Y4=	-.043661434135	-.043661434135	.3469E-16
H= .700	Y= .932393817961		
Y1=	-4798.251805979109	-4798.251805979105	.4547E-11
Y2=	-484.231688967515	-484.231688967515	.3979E-12
Y3=	4.358085200708	4.358085200708	-.4441E-14
Y4=	-.048423169618	-.048423169618	.4163E-16
H= .800	Y= .923116344186		
Y1=	-5216.399122165437	-5216.399122165433	.3638E-11
Y2=	-526.430429121550	-526.430429121549	.4547E-12
Y3=	4.737873862094	4.737873862094	-.3553E-14
Y4=	-.052643043697	-.052643043697	.4857E-16
H= .900	Y= .913931182820		
Y1=	-5586.050438383845	-5586.050438383841	.4547E-11
Y2=	-563.735032635366	-563.735032635366	.4547E-12
Y3=	5.073615293718	5.073615293718	-.3553E-14
Y4=	-.056373504104	-.056373504104	.4163E-16
H= 1.000	Y= .904837415339		
Y1=	-5911.907365731702	-5911.907365731698	.4547E-11
Y2=	-596.619978376395	-596.619978376395	.4547E-12
Y3=	5.369579805388	5.369579805388	-.4441E-14
Y4=	-.059661998727	-.059661998727	.4857E-16

```
RESULT FOR ORDER 4 h=0.1 ON PROBLEM 2

V(I)=  -.1101E+05   -.1111E+04   .1000E+02   -.1111E+00

W(I)=   .1101E+05    .1111E+04  -.1000E+02    .1111E+00

A R RQ   .100067E+01    .118993E+00    .980199E+00

A R RQ   .100666E+01    .123295E+00    .818731E+00

YP(I)=   .980198672723   .818730750638

H=  .200     Y=     .980198672723

Y1= -1777.761825094168   -1777.761825094169   -.1137E-11

Y2=  -179.408802613197    -179.408802613197   -.1137E-12

Y3=     1.614679223519       1.614679223519    .1110E-14

Y4=     -.017940880529       -.017940880529   -.1041E-16

H=  .400     Y=     .960789438007

Y1= -3198.068054888935   -3198.068054888937   -.2274E-11

Y2=  -322.743773830753    -322.743773830754   -.2842E-12

Y3=     2.904693964477       2.904693964477    .2220E-14

Y4=     -.032274377864       -.032274377864   -.2082E-16

H=  .600     Y=     .941764531900

Y1= -4326.411444013019   -4326.411444013021   -.1819E-11

Y2=  -436.614334848423    -436.614334848423   -.2274E-12

Y3=     3.929529013636       3.929529013636    .2220E-14

Y4=     -.043661434135       -.043661434135   -.2082E-16

H=  .800     Y=     .923116344186

Y1= -5216.399122165430   -5216.399122165433   -.3638E-11

Y2=  -526.430429121549    -526.430429121549   -.3411E-12

Y3=     4.737873862094       4.737873862094    .3553E-14

Y4=     -.052643043697       -.052643043697   -.3469E-16

H= 1.000     Y=     .904837415339

Y1= -5911.907365731694   -5911.907365731698   -.3638E-11

Y2=  -596.619978376395    -596.619978376395   -.3411E-12

Y3=     5.369579805388       5.369579805388    .2665E-14

Y4=     -.059661998727       -.059661998727   -.3469E-16
```



```
WRITE(3,70)Y2N(I),EXY2(I),ER2
```

```
      WRITE(3,90)Y3N(I),EXY3(I),ER3

      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SL1', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-1.0
      P=D(1)-D(2)
      V(1)=(-9909.0-D(2))/P
      V(2)=(-1000.0-D(2))/P
      V(3)=(9.0-D(2))/P
      V(4)=(-0.1-D(2))/P
      W(1)=(9909.0+D(1))/P
      W(2)=(1000.0+D(1))/P
      W(3)=(-9.0+D(1))/P
      W(4)=(0.1+D(1))/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.005
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 7
C     EXPONENTIAL FUNCTION IS USED FOR PROBLEM 2
      RQN=15.+7.*Q+Q**2+16.*Q*EXP(Q)
      RQD=15.-7.*Q+Q**2
      RQ=RQN/RQD
      WRITE(3,10)Q,
      WRITE(3,11)RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T1=RQ**0.5
      T=RQ**1.5
      YN1=1.+1049.*Q/1890.+37.*(Q**2)/315.+2.*T/945.+T1*(34.*Q/35
     1.+4.*(Q**2)/105.)
      YD=1.-33.*Q/70.+8.*(Q**2)/105.
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,20
      H=H+0.01
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+0.01
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
```

```
      WRITE(3,80)Y3N(I),EXY3(I),ER3
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 7 h=0.005 ON PROBLEM 2',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'Q = ',2F10.4,5X,2X/)
   11 FORMAT(5X 'RQ = ',E15.6,2X/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y=',F18.12,/)
   65 FORMAT(5X,'Y1=',F18.12,5X,F18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2=',F18.12,5X,F18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3=',F18.12,5X,F18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4=',F18.12,5X,F18.12,5X,E10.4,/)
  100 STOP
      END
```

```
RESULT FOR ORDER 7 h=0.05 ON PROBLEM 2

V(I)=  -.1101E+05    -.1111E+04    .1000E+02    -.1111E+00

W(I)=   .1101E+05     .1111E+04   -.1000E+02     .1111E+00

RQ    .990050E+00

RQ    .904837E+00

YP(I)=      .992140220779      .906706869786

H=  .100      Y=      .992140220779

Y1= -940.621195996117      -938.188710152108      .2432E+01

Y2=  -94.925945705532       -94.680463230609      .2455E+00

Y3=     .854333511350          .852124169075     -.2209E-02

Y4=    -.009492594712         -.009468046464      .2455E-04

H=  .200      Y=      .984342217688

Y1=-1786.095821341269     -1777.761825094169      .8334E+01

Y2= -180.249855822108      -179.408802613197      .8411E+00

Y3=    1.622248702399         1.614679223519     -.7569E-02

Y4=    -.018024985851         -.017940880529      .8411E-04

H=  .300      Y=      .976605505180

Y1=-2545.358505377342     -2528.196745758464      .1716E+02

Y2= -256.873398463754      -255.141461878945      .1732E+01

Y3=    2.311860586174         2.296273156911     -.1559E-01

Y4=    -.025687340229         -.025514146568      .1732E-03

H=  .400      Y=      .968929601523

Y1=-3226.509881192059     -3198.068054888937      .2844E+02

Y2= -325.614076212742      -322.743773830754      .2870E+01

Y3=    2.930526685915         2.904693964477     -.2583E-01

Y4=    -.032561408106         -.032274377864      .2870E-03

H=  .500      Y=      .961314028775

Y1=-3836.894395329803     -3795.133440748522      .4176E+02

Y2= -387.213078547765      -382.998631622618      .4214E+01

Y3=    3.484917706930         3.446987684604     -.3793E-01

Y4=    -.038721308432         -.038299863733      .4214E-03

H=  .600      Y=      .953758312747
```

Y1=	-4383.170858362740	-4326.411444013021	.5676E+02
Y2=	-442.342401691668	-436.614334848423	.5728E+01
Y3=	3.981081615225	3.929529013636	-.5155E-01
Y4=	-.044234240828	-.043661434135	.5728E-03
H= .700	Y= .946261982979		
Y1=	-4871.376413550332	-4798.251805979105	.7312E+02
Y2=	-491.611304223467	-484.231688967515	.7380E+01
Y3=	4.424501738011	4.358085200708	-.6642E-01
Y4=	-.049161131155	-.048423169618	.7380E-03
H= .800	Y= .938824572708		
Y1=	-5306.984537634598	-5216.399122165433	.9059E+02
Y2=	-535.572160423312	-526.430429121549	.9142E+01
Y3=	4.820149443810	4.737873862094	-.8228E-01
Y4=	-.053557216840	-.052643043697	.9142E-03
H= .900	Y= .931445618840		
Y1=	-5694.957630532069	-5586.050438383841	.1089E+03
Y2=	-574.725767537801	-563.735032635366	.1099E+02
Y3=	5.172531907840	5.073615293718	-.9892E-01
Y4=	-.057472577610	-.056373504104	.1099E-02
H= 1.000	Y= .924124661920		
Y1=	-6039.794698741433	-5911.907365731698	.1279E+03
Y2=	-609.526157911135	-596.619978376395	.1291E+02
Y3=	5.485735421200	5.369579805388	-.1162E+00
Y4=	-.060952616699	-.059661998727	.1291E-02

APPENDIX C

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 3

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RS1', STATUS = 'NEW')
      A1=1.0
      C1=-1.0
      D=-1.0
      HH=0.05
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MU
C     ISTEP METHOD FOR THE CASE K=2 ORDER 3 PROBLEM 3
      APN=(EXP(2.*Q)-1.)/Q-2.-Q*(EXP(2.*Q)+1.)
      APD=-1.-Q+(1.-Q)*EXP(2.*Q)
      A=APN/APD
      RPN=(EXP(2.*Q)-1.)/Q-1.-EXP(2.*Q)+Q*(EXP(2.*Q)-1.)/3.
      RPD=EXP(3.*Q)+EXP(2.*Q)*(4.+5.*Q)/4.-(8.+15.*Q)/4.
      R=RPN/RPD
      RQN=1.0+Q*(2.0-A)+(Q**2)*(1.0-A)
      RQD=1.0-A*Q-(1.0-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      T=RQ**1.5
      YN1=4.0*R*Q*T-R*Q*(8.+15.*Q)+4.*(1.+Q+(Q**2)/3.)
      YD=4.*(1.-Q+(Q**2)/3.)-R*Q*(4.+5.*Q)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=-1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.1
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.1
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
 30   CONTINUE
 99   FORMAT(5X,'RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 3 USING
     1=0.05',/)
 20   FORMAT(5X,'YN2= ',E18.12/)
 10   FORMAT(5X,'A R RQ  ',3(E12.6,2X)/)
 40   FORMAT(5X,'H= ',F6.4,5X,'Y= ',F18.12,/)
 65   FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
 70   FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
 85   FORMAT(25X,'Q= ',F7.4,2X/)
 100  STOP
      END
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 3 USING h = 0.05

Q= 0500

A B RQ .190000E+02 .255324E-05 .904547E+00

YN2= .904837416637E+00

H= .1000 Y= .904837416637

YN=	.904837416637	.904837416688	.5040E-10
ZN=	.904837416637	-.904837416688	-.5040E-10

H= .2000 Y= .818730750547

YN=	.818730750547	.818730750638	.9121E-10
ZN=	.818730750547	-.818730750638	-.9121E-10

H= .3000 Y= .740818217246

YN=	.740818217246	.740818217370	.1238E-09
ZN=	.740818217246	-.740818217370	-.1238E-09

H= .4000 Y= .670320041891

YN=	.670320041891	.670320042040	.1494E-09
ZN=	.670320041891	.670320042040	-.1494E-09

H= .5000 Y= .606530655025

YN=	.606530655025	.606530655194	.1689E-09
ZN=	.606530655025	-.606530655194	-.1689E-09

H= .6000 Y= .548811631004

YN=	.548811631004	.548811631187	.1834E-09
ZN=	.548811631004	-.548811631187	-.1834E-09

H= .7000 Y= .496585298418

YN=	.496585298418	.496585298612	.1936E-09
ZN=	.496585298418	-.496585298612	-.1936E-09

H= .8000 Y= .449328958561

YN=	.449328958561	.449328958761	.2002E-09
ZN=	.449328958561	-.449328958761	.2002E-09

H= .9000	Y=	.406569654084	
YN=	.406569654084	.406569654288	.2038E-09
ZN=	.406569654084	-.406569654288	-.2038E-09
H= 1.0000	Y=	.367879435485	
YN=	.367879435485	.367879435690	.2049E-09
ZN=	.367879435485	-.367879435690	-.2049E-09

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='SOL1', STATUS = 'NEW')
      A1=1.0
      C1=-1.0
      D=-1.0
      HH=0.1
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 PROBLEM 3
      APN=1.+2.*Q+4.*(Q**2)/3.+EXP(2.*Q)*(2.*(Q**2)-3.)/3.
      APD=Q*EXP(2.*Q)*(Q-1.)+Q*(1.+Q)
      A=APN/APD
      RPN=1.+Q+(Q**2)/3.-EXP(2.*Q)*(1.-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2*Q)+1)/4.0-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.0+Q*(2.0-A)+(Q**2)*(4.0/3.-A)
      RQD=1.0-Q*A-(Q**2)*(2./3.-A)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      T=RQ**1.5
      YN1=R*Q*T+1.+Q*(1.-R)+(Q**2)*(1./3.-(3./4.)*R)
      YD=1.-Q+(Q**2)*(1./3.+(3./4.)*R)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=-1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.2
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.2
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
 30   CONTINUE
 99   FORMAT(5X,'RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING H
     1=0.1',/)
 20   FORMAT(5X,'YN2= ',E18.12)
 10   FORMAT(5X,'A R RQ '3(E12.6,2X)/)
 40   FORMAT(5X,'H= ',F6.4,5X,'Y= ',F18.12,/)
 65   FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
 70   FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
 85   FORMAT(25X,'Q= ',F6.2,2X/)
100   STOP
      END
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING h =0.0027

Q= .00

A E RQ .100019E101 .118102E100 .994615E100

YN2= .994614553652E100

H= .0054 Y= .994614553652

YN= .994614553652 .994614553652 .0000E100

ZN= .994614553652 -.994614553652 .0000E100

H= .0108 Y= .989258110337

YN= .989258110337 .989258110337 -.1110E 15

ZN= .989258110337 -.989258110337 .1110E 15

H= .0162 Y= .983930513859

YN= .983930513859 .983930513859 -.1110E 15

ZN= -.983930513859 -.983930513859 .1110E 15

H= .0216 Y= .978631608867

YN= .978631608867 .978631608867 -.1110E 15

ZN= .978631608867 -.978631608867 .1110E 15

H= .0270 Y= .973361240843

YN= .973361240843 .973361240843 -.2220E 15

ZN= .973361240843 .973361240843 .2220E 15

H= .0324 Y= .968119256103

YN= .968119256103 .968119256103 -.2220E 15

ZN= .968119256103 .968119256103 .2220E 15

H= .0378 Y= .962905501791

YN= .962905501791 .962905501791 .2220E 15

ZN= .962905501791 -.962905501791 .2220E 15

H= .0432 Y= .957719825873

YN= .957719825873 .957719825873 .3331E 15

ZN= .957719825873 -.957719825873 .3331E 15

H= .0486 Y= .952562077134

YN= .952562077131 .952562077131 .3331E-15

ZN= .952562077134 -.952562077134 .3331E-15

H= .0540 Y= .947432105175

YN= .947432105175 .947432105175 .4441E-15

ZN= .947432105175 .947432105175 .4441E-15

H= .0594 Y= .942329760404

YN= .942329760404 .942329760404 .4441E-15

ZN= -.942329760404 .942329760404 .4441E-15

H= .0648 Y= .937254894037

YN= .937254894037 .937254894037 -.3331E-15

ZN= -.937254894037 -.937254894037 .3331E-15

H= .0702 Y= .932207358091

YN= .932207358091 .932207358091 -.3331E-15

ZN= .932207358091 -.932207358091 .3331E-15

H= .0756 Y= .927187005379

YN= .927187005379 .927187005379 -.4441E-15

ZN= -.927187005379 -.927187005379 .4441E-15

H= .0810 Y= .922193689507

YN= .922193689507 .922193689507 .4441E-15

ZN= .922193689507 -.922193689507 .4441E-15

[illegible]

YN= .917227264870 .917227264870 -.5551E-15

ZN= .917227264830 -.917227264870 .5551E-15

YN= .912287586659 .912287586616 .1276E-10

ZN= .912287586659 -.912287586616 .1276E-10

```
H=   .0972      Y=        .907374510808
YN=          .827786502649       -.827786502637       .1226E-10
ZN=          .907374510808       -.907374510794       .1344E-10
H=   .1026      Y=        .902487894063
YN=          .823328502852       -.823328502839       .1287E-10
ZN=          .818894511373       -.818894511360       .1280E-10
H=   .1080      Y= 360     .4441E-15
H=   .1080      Y=        .897627593916
YN=          .818391511360       -.835069259634       -.1617E 01
ZN=          .814484398905       -.830572038939       .1609E 01
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING h = 0.05

Q= .05

A R RQ .100503E101 .120398E100 .904837E100

YN2= .904837416688E100

H= .1000	Y= .904837416688		
YN=	.904837416688	.904837416688	.0000E100
ZN=	.904837416688	-.904837416688	.0000E100
H= .2000	Y= .818730750638		
YN=	.818730750638	.818730750638	.0000E100
ZN=	.818730750638	-.818730750638	.0000E100
H= .3000	Y= .740818217370		
YN=	.740818217370	.740818217370	.0000E100
ZN=	.740818217370	.740818217370	.0000E100
H= .4000	Y= .670320042040		
YN=	.670320042040	.670320042040	.0000E100
ZN=	.670320042040	.670320042040	.0000E100
H= .5000	Y= .606530655194		
YN=	.606530655194	.606530655194	.1110E-15
ZN=	.606530655194	-.606530655194	-.1110E-15
H= .6000	Y= .548811631187		
YN=	.548811631187	.548811631187	.1110E-15
ZN=	.548811631187	.548811631187	.1110E-15
H= .7000	Y= .496585298612		
YN=	.496585298612	.496585298612	.1665E-15
ZN=	.496585298612	-.496585298612	.1665E-15
H= .8000	Y= .449328958761		
YN=	.449328958761	.449328958761	.1665E-15
ZN=	.449328958761	.449328958761	.1665E-15

```
H=   .9000      Y=        .406569654288

YN=        .406569654288        .406569654288        .1665E-15

ZN=        .406569654288       -.406569654288       -.1665E-15

H=  1.0000      Y=        .367879435690

YN=        .367879435690        .367879435690        .1665E-15

ZN=        .367879435690       -.367879435690       -.1665E-15
```

```
RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING h = 0.1

                         Q=     -.10

A R RQ   .100666E+01      .123295E+00     .818731E+00

YN2=   .818730750638E+00

H=   .2000        Y=          .818730750638

YN=           .818730750638         .818730750638           .1110E-15

ZN=          -.818730750638        -.818730750638           .1110E-15

H=   .4000        Y=          .670320042040

YN=           .670320042040         .670320042040          -.2220E-15

ZN=           .670320042040        -.670320042040           .2220E-15

H=   .6000        Y=          .548811631187

YN=           .548811631187         .548811631187          -.2220E-15

ZN=           .548811631187        -.548811631187           .2220E-15

H=   .8000        Y=          .449328958761

YN=           .449328958761         .449328958761          -.2220E-15

ZN=           .449328958761         .449328958761           .2220E-15

H=  1.0000        Y=          .367879435690

YN=           .367879435690         .367879435690          -.2220E-15

ZN=           .367879435690        -.367879435690           .2220E-15
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='WAS1', STATUS = 'NEW')
      A1=1.0
      C1=-1.0
      D=-1.0
      HH=0.0027
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 6 PROBLEM 3
      APN=3.0+14.0*Q/5.+(2./3.)*(Q**2)+(4.*Q/15.)*EXP(Q)*(12.0+7.
     10*Q)+EXP(2.0*Q)*(4.*(Q**2)-45.0)/15.0
      APD=Q*(3.0*Q)+4.*(Q**2)*EXP(Q)-Q*EXP(2.*Q)*(3.-Q)
      A=APN/APD
      RPN=1.+7.*Q/15.+(Q**2)/15.+16.*Q*EXP(Q)/15.-EXP(2.*Q)*(1.0-
     17.*Q/15.+(Q**2)/15.0)
      RPD=Q*(10.+3.*Q)+9.*Q*EXP(Q)*(1.+2.*Q)-9*Q*EXP(2.*Q)*(2.-Q)
     1-Q*EXP(3.0*Q)
      R=RPN/RPD
      RQN=1.0+Q*(1.5-A)+(Q**2)*(41./45.-7.*A/6.)+(Q**3)*(.2-.5*A)
      RQD=1.0-Q*(A+0.5)-(Q**2)*(4.0/45.-5.0*A/6.0)+(Q**3)*(2./45
     1.0-A/6.0)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,10)A,R,RQ
      S=RQ**0.5
      T=RQ**1.5
      YN1=1.0+(7./15.-10.0*R)*Q+(1./15.-3.*R)*(Q**2)+R*Q*T+S*((1
     16./15.-9.*R)*Q-18.*R*(Q**2))
      YD=1.-(7/15.+18.*R)*Q+(1./15.+9.*R)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=-1.0
      H=0.0
      X=0.0
      DO 30 I=1,20
      H=H+0.0054
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.0054
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
 30   CONTINUE
 99   FORMAT(5X,'RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 6 USING h
     1=0.0027',/)
 20   FORMAT(5X,'YN2',F18.12)
 10   FORMAT(5X,'A R RQ',3(E12.6,2X)/)
 40   FORMAT(5X,'H= ',F6.4,5X,'Y= ',F18.12,/)
 65   FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
 70   FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
 100  STOP
      END
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 6 USING h =0.0027

A = -.100743E-13   R = -.453438E+03   RQ = .994615E+00

YN2   .994614553653

H= .0054	Y=	.994614553653	
YN=	.994614553653	.994614553652	-.8182E-12
ZN=	.994614553653	-.994614553652	.8182E-12
H= .0108	Y=	.989258110338	
YN=	.989258110338	.989258110337	-.1628E-11
ZN=	-.989258110338	-.989258110337	.1628E-11
H= .0162	Y=	.983930513861	
YN=	.983930513861	.983930513859	.2428E-11
ZN=	.983930513861	-.983930513859	.2428E-11
H= .0216	Y=	.978631608870	
YN=	.978631608870	.978631608867	.3220E-11
ZN=	.978631608870	-.978631608867	.3220E-11
H= .0270	Y=	.973361240847	
YN=	.973361240847	.973361240843	.4004E-11
ZN=	.973361240847	-.973361240843	.4004E-11
H= .0324	Y=	.968119256108	
YN=	.968119256108	.968119256103	.4779E-11
ZN=	.968119256108	-.968119256103	.4779E-11
H= .0378	Y=	.962905501796	
YN=	.962905501796	.962905501791	.5515E-11
ZN=	.962905501796	.962905501791	.5515E-11
H= .0432	Y=	.957719825879	
YN=	.957719825879	.957719825873	.6303E-11
ZN=	.957719825879	.957719825873	.6303E-11

H= .0486	Y= .952562077141		
YN=	.952562077141	.952562077134	.7053E-11
ZN=	.952562077141	.952562077134	.7053E-11
H= .0540	Y= .947432105183		
YN=	.947432105183	.947432105175	-.7794E-11
ZN=	.947432105183	.947432105175	.7794E-11
H= .0594	Y= .942329760413		
YN=	.942329760413	.942329760404	.8528E-11
ZN=	.942329760413	.942329760404	.8528E-11
H= .0648	Y= .937254894047		
YN=	.937254894047	.937254894037	-.9253E-11
ZN=	.937254894047	-.937254894037	.9253E-11
H= .0702	Y= .932207358101		
YN=	.932207358101	.932207358091	.9970E-11
ZN=	.932207358101	-.932207358091	.9970E-11
H= .0756	Y= .927187005390		
YN=	.927187005390	.927187005379	-.1068E-10
ZN=	.927187005390	-.927187005379	.1068E-10
H= .0810	Y= .922193689518		
YN=	.922193689518	.922193689507	-.1138E-10
ZN=	.922193689518	-.922193689507	.1138E-10
H= .0864	Y= .917227264882		
YN=	.917227264882	.917227264870	-.1207E-10
ZN=	.917227264882	.917227264870	.1207E-10
H= .0918	Y= .912287586659		
YN=	.912287586659	.912287586646	.1276E-10
ZN=	.912287586659	.912287586646	.1276E-10

H= .0972	Y=	.907374510808	
YN=	.827786502649	.827786502637	.1220E-10
ZN=	.907374510808	.907374510794	.1341E-10
H= .1026	Y=	.902487894063	
YN=	.823328502852	.823328502839	.1287E-10
ZN=	.818894511373	.818894511360	.1280E-10

RESULT OF PROBLEM 3 FOR CASE K-2 ORDER 6 USING L = 0.05

A = .119185E-08   R = .429431E 02   RQ = .904838E+00

YN2   .904837402983

H= .1000   Y= .904837402983

YN= .904837402983   .904837416688   .1370E 07

ZN= .904837402983   .904837416688   .1370E 07

H= .2000   Y= .818730725836

YN= .818730725836   .818730750638   .2480E 07

ZN= .818730725836   -.818730750638   -.2480E-07

H= .3000   Y= .740818183708

YN= .740818183708   .740818217370   -.3366E-07

ZN= -.740818183708   -.740818217370   -.3366E-07

H= .4000   Y= .670320001429

YN= .670320001429   .670320042040   .4061E 07

ZN= -.670320001429   -.670320042040   -.4061E 07

H= .5000   Y= .606530609260

YN= .606530609260   .606530655191   .4593E-07

ZN= .606530609260   .606530655191   .4593E-07

H= .6000   Y= .548811581312

YN= .548811581312   .548811631187   .4987E 07

ZN= .548811581312   -.548811631187   -.4987E 07

H= .7000   Y= .496585245962

YN= .496585245962   .496585298612   .5265E-07

ZN .496585245962   .496585298612   .5265E 07

H= .8000   Y= .449328904317

YN .449328904315   .449328958761   .5445E 07

ZN= .449328904315   .449328958761   .5445E 07

```
H=  .9000      Y=      .406569598866

YN        .406569598866        .406569654288      .5542E-07

ZN=       .406569598866        .406569654288      .5512E-07

H= 1.0000      Y=      .367879379969

YN        .367879379969        .367879435690      .5572E-07

ZN=       .367879379969        .367879435690      .5572E-07
```

APPENDIX D

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 4

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16),EXZZ(16),D(6),Q1(
     16),C(6),ZZN(16),V(4),YP(4)
      OPEN(UNIT = 3, FILE='RST2', STATUS = 'NEW')
      D(1)=0.0
      D(2)=-0.00928572
      D(3)=-3500.003714
      V(1)=(D(2)+0.013)/D(2)
      V(2)=1.0+32.5/(D(2)*D(3))
      V(3)=(0.013*(D(3)-D(2))+45.5002)/(D(2)*D(3))
      B1=(13.00002+0.013*D(3))/(D(2)*(D(2)-D(3)))
      B2=-32.5/(D(2)*(D(3)-D(2)))
      B3=(0.013*D(3)-45.5002)/(D(2)*(D(3)-D(2)))
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),B1,B2,B3
      HH=0.1
      DO 31 K=1,3
      Q1(K)=D(K)*HH
      Q=Q1(K)
      IF(Q.EQ.0.0)GO TO 44
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 2 ON PROBLEM 4
      APN=(1.0-EXP(2.*Q))/Q+2.0
      APD=1.-EXP(2.0*Q)
      A=APN/APD
      RPN=(EXP(2.*Q)-1.)/Q-EXP(2.*Q)-1.0
      RPD=EXP(3.*Q)-3.*EXP(2.*Q)/2.+0.5
      R=RPN/RPD
      RQN=1.+(2.-A)*Q
      RQD=1.-A*Q
      RQ=RQN/RQD
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+0.5*Q*(2.+R)
      YD=1.-0.5*Q*(2.-3.*R)
      YP(K)=YN1/YD
   44 C(K)=V(K)
   31 CONTINUE
      WRITE(3,50)C(1),C(2),C(3)
      WRITE(3,20)YP(1),YP(2),YP(3)
      Y=1.0
      Z=1.0
      ZZ=0.0
      H=0.0
      X=0.0
      DO 30 I=1,5
      H=H+0.2
      Y=Y*YP(2)
      YN(I)=V(1)+Y*B1
      ZN(I)=V(2)+Y*B2
      ZZN(I)=V(3)+Y*B3
      WRITE(3,40)H,Y
      X=X+0.2
      EXY(I)=V(1)+B1*EXP(D(2)*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=V(2)+B2*EXP(D(2)*X)
      ERZ=EXZ(I)-ZN(I)
      EXZZ(I)=V(3)+B3*EXP(D(2)*X)
      ERZZ=EXZZ(I)-ZZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
      WRITE(3,80)ZZN(I),EXZZ(I),ERZZ
```

```
 99 FORMAT(5X,'RESULT FOR ORDER 2 h=0.1 ON PROBLEM 4 ',/)
 90 FORMAT(5X,'V(I) B',6(F8.4,2X)/)
 50 FORMAT(5X,'C(I)=',3(F6.4,2X)/)
 20 FORMAT(5X,'YP(I) ='3(E10.4,2X)/)
 10 FORMAT(5X,'A R RQ',3(E18.12,2X)/)
 40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
 65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
 70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
 80 FORMAT(5X,'ZZ= ',F18.12,5X,F15.12,5X,E10.4,/)
100 STOP
    END
```

```
RESULT FOR ORDER 2 h=0.0625 ON PROBLEM 4
Y(I) B    .1000     2.0000      .0000     1.0000     1.0000     2.0...
A R RQ .100019345253E+01    .444610596177E+00    .998839958361E...
S R RQ .199512857621E+01  -.199085715212E+01    .229793622073E...
Y(I) = .4000   2.0000    .0000
YP(I) = .0000E+00    .9900E+00    .1152E 10
H=    .125       Y=        .998839958361
YN=          .598812660537          .598812660537         .0000E+00
ZN=         1.001157369718         1.001157369718         .0000E+00
ZZ=         2.796753900738         2.796753900738         .0000E+00
H=    .250       Y=        .997681262419
YN=          .597683962437          .597683962437        -.1110E-15
ZN=         1.002316086801         1.002316086801         .2220E-15
ZZ=         2.793509540102         2.793509540102         .4441E-15
H=    .375       Y=        .996523910612
YN=          .596526608475          .596526608475         .1110E-15
ZN=         1.003473439747         1.003473439747         .2220E-15
ZZ=        -2.790268943061        -2.790268943061         .4441E-15
H=    .500       Y=        .995367901381
YN=          .595370597092          .595370597092         .2220E-15
ZN=         1.004629150116         1.004629150116         .2220E-15
ZZ=         2.787032105246         2.787032105246         .8882E-15
H=    .625       Y=        .994213233169
YN=          .594215926731          .594215926731         .3331E-15
ZN=         1.005784119161         1.005784119161         .2220E-15
ZZ=         2.783799022299        -2.783799022299         .8882E-15
```

H= .750	Y= .993059904421		
YN=	.593062595835	.593062595835	.3331E-15
ZN=	1.006937449348	1.006937449348	.4441E-15
ZN=	2.780569689862	2.780569689862	.8882E-15
H= .875	Y= .991907913581		
YN=	.591910602851	.591910602851	.4441E-15
ZN=	1.008089141321	1.008089111321	.4441E-15
ZZ=	2.777311103585	2.777311103585	.1332E-14
H= 1.000	Y= .990757259100		
YN=	.590759916227	.590759916227	.4441E-15
ZN=	1.009240096936	1.009240096936	.4441E-15
ZZ=	2.774122259123	2.774122259123	.1332E-14

```
RESULT FOR ORDER 2 h=0.1 ON PROBLEM 4
X(I) =   .1000    2.0000     .0000   1.0000    1.0000    2.0000
X R RQ .106030952399E+01   .444719629869E+00   .998111579384E+00
Y R RQ .199571120017E+01   .199128572036E+01   .130205502992E-16
C(I)=  .1000   2.0000    .0000
YP(I)    .0000E+00    .9981E+00    .9025E-17
H=  .200        Y=         .598144579384
YN=        .598147280265        .598147280265        .1110E-15
ZN=       1.001852769379       1.001852769379        .2220E-15
ZZ=       2.794806832403       2.794806832403        .4441E-15
H=  .400        Y=         .596292601351
YN=        .596295298787        .596295298787        .2220E-15
ZN=       1.003704749233       1.003704749233        .2220E-15
ZZ=       2.789621274715       2.789621274715        .4441E-15
H=  .600        Y=         .594414059522
YN=        .594446753514        .594446753514        .3331E-15
ZN=       1.005553292884       1.005553292884        .4441E-15
ZZ=       2.784445338477       2.784445338477        .8882E-15
H=  .800        Y=         .592598947513
YN=        .592601638069        .592601638069        .5551E-15
ZN=       1.007398106709       1.007398106709        .6661E-15
ZZ=       2.779279005748       2.779279005748        .1332E-14
H= 1.000        Y=         .590757258963
YN=        .590759946090        .590759946090        .6661E-15
ZN=       1.009210097073       1.009210097073        .6661E-15
ZZ=       2.774122258739       2.774122258739        .1776E-14
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16),EXZZ(16),D(6),Q1(
     16),C(6),ZZN(16),V(4),YP(4)
      OPEN(UNIT = 3, FILE='TAL', STATUS = 'NEW')
      D(1)=0.0
      D(2)=-0.00928572
      D(3)=-3500.003714
      V(1)=(D(2)+0.013)/D(2)
      V(2)=1.0+32.5/(D(2)*D(3))
      V(3)=(0.013*(D(3)-D(2))+45.5002)/(D(2)*D(3))
      B1=(13.00002+0.013*D(3))/(D(2)*(D(2)-D(3)))
      B2=-32.5/(D(2)*(D(3)-D(2)))
      B3=(0.013*D(3)-45.5002)/(D(2)*(D(3)-D(2)))
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),B1,B2,B3
      HH=0.0625
      DO 31 K=1,3
      Q1(K)=D(K)*HH
      Q=Q1(K)
      IF(Q.EQ.0.0)GO TO 44
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 OF PROBLEM 4
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YP(K)=YN1/YD
   44 C(K)=V(K)
   31 CONTINUE
      WRITE(3,50)C(1),C(2),C(3)
      WRITE(3,20)YP(1),YP(2),YP(3)
      Y=1.0
      Z=1.0
      ZZ=0.0
      H=0.0
      X=0.0
      DO 30 I=1,8
      H=H+0.125
      Y=Y*YP(2)
      YN(I)=V(1)+Y*B1
      ZN(I)=V(2)+Y*B2
      ZZN(I)=V(3)+Y*B3
      WRITE(3,40)H,Y
      X=X+0.125
      EXY(I)=V(1)+B1*EXP(D(2)*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=V(2)+B2*EXP(D(2)*X)
      ERZ=EXZ(I)-ZN(I)
      EXZZ(I)=V(3)+B3*EXP(D(2)*X)
      ERZZ=EXZZ(I)-ZZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
```

```
      WRITE(3,80)ZZN(I),ERZZ(I),ERZZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PRB 4 FOR ORDER 4 h=0.0625',/)
   90 FORMAT(5X,'V(I) B',6(F8.4,2X)/)
   50 FORMAT(5X,'C(I)=',3(F6.4,2X)/)
   20 FORMAT(5X,'YP(K) ',3(F18.12,2X)/)
   10 FORMAT(5X,'A R RQ',3(E18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   80 FORMAT(5X,'ZZ= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(5X,'Q = ',F12.6,2X/)
  100 STOP
      END
```

```
RESULT OF PROBLEM 4 FOR ORDER 4 L=0.0625
V(I) B   .1000   2.0000   .0000   1.0000   1.0000   2.0000
Q =      .000580
A R EQ .99970901390OE100   .893235908462E100   .998839958361E100
Q =     218.750229
A R EQ .133029271541E101   .441065472874E100   .706390904169E-16
C(I)= .1000  2.0000   .0000
YP(K)  .0000E100   .9938E100   .2812E 17
H   .125     Y=        .998839958361
YR=       .598812660537        .598812660537        .1110E 15
YR       1.001157389718       1.001157389718        .2220E 15
ZZ=      2.796753900738       2.796753900738        .1411E 15
H   .250     Y=        .997681262419
YR=       .597683962137        .597683962137        .1110E 15
YR       1.002316086001       1.002316086001        .0000E100
ZZ=      2.793509510102       2.793509510102        .1111E 15
H   .375     Y         .996522910612
YR=       .596526608475        .596526608475        .2220E 15
ZR=      1.003473439747       1.003473439747        .2220E 15
ZZ=      2.790268943061       2.790268943061        .1111E 15
H   .500     Y         .995367901381
YR=       .595370597092        .595370597092        .2220E 15
YR=      1.004629150116       1.004629150116        .2220E 15
ZZ=      2.787032105246       2.787032105246        .1111E 15
H   .625     Y         .994219223164
YR=       .594215926791        .594215926791        .2220E 15
YR       1.005784119164       1.005784119171        .2220E 15
ZZ=      2.783799022299       2.783799022299        .1111E 15
```

```
H=   .750       Y=         .993059904421

YN          .593062595835          .593062595835        .3331E 15

ZN=        1.006937449348         1.006937449348        .2220E 15

XN=        2.780569689862         2.780569689862        .8882E 15

H=   .875       Y=         .991907913581

YN=         .591910602851          .591910602851        .3331E 15

ZN=        1.008039411321         1.008039411321        .2220E 15

XN=        2.777314103585         2.777314103585        .8882E 15

H=  1.000       Y=         .990757259100

YN=         .590759916227          .590759940227        .4441E 15

ZN=        1.009240096936         1.009240096936        .4441E 15

XN=        2.774122259123         2.774122259123        .8882E 15
```

```
RESULT OF PROBLEM 4 FOR ORDER 4 h=0.1
V(I) B    .1000    2.0000     .0000    1.0000    -1.0000    2.0
Q =      .000929
A R RQ .100005272217E+01   .100631094613E+00   .998144579384E+
Q =    350.000371
A R RQ .133143130233E+01 . .442330877229E+00  -.273108660792E
C(L)= .1000   2.0000    .0000
YP(K)   .0000E+00    .9981E+00   -.2556E-16
H=   .200      Y=      .998144579384
YH=        .598147280265        .598147280265      -.1110E-15
ZH=       1.001852769379       1.001852769379       .2220E-15
ZZ=       2.794806832403       2.794806832403       .4111E-15
H=   .400      Y=      .996292601354
YH=        .596295298787        .596295298787       .2220E-15
ZH=       1.003704749233       1.003704749233       .2220E-15
ZZ=       2.789621274745       2.789621274745       .4441E-15
H    .600      Y=      .991144059522
YH=        .594446753514        .594446753514       .3331E-15
ZH=       1.005553292884       1.005553292884       .4441E-15
ZZ=       2.784415338477       2.784415338477       .8882E-15
H    .800      Y=      .992598917513
YH=        .592601638069        .592601638069       .5551E-15
ZH=       1.007398406709       1.007398406709       .6661E-15
ZZ=       2.779279065748       2.779279065748       .1332E-14
H   1.000      Y=      .990757258963
YH=        .590759946090        .590759946090       .6661E-15
ZH=       1.009240097073       1.009240097073       .6661E-15
ZZ=       2.774122258739       2.774122258739       .1776E-14
```

RESULT OF PROBLEM 4 FOR ORDER 4 h=0.2

V(I) B .1000 2.0000 .0000 1.0000 1.0000 -2.000

Q - .001857

A R Eq .10001243323 7E+01 .1191086676 04E+00 .9962920135 4E+00

Q - 700 .000743

A R Eq .1322381634 61E+01 .4139869515 21E+00 .1925358726 17E 1

C(I) .1000 2.0000 .0000

YP(E) .0000E+00 .9963E+00 .1014E-16

H .100 Y= .996292013 51

YN=	.596295298787	.596295298787	.0000E+00
ZN=	1.003701719233	1.003701719233	.0000E+00
ZZ=	2.789621274745	2.789621274745	.0000E+00

H= .800 Y= .992598917513,

YN=	.592601638069	.592601638069	.0000E+00
ZN=	1.007393406709	1.007393406709	.0000E+00
ZZ=	2.779279005748	2.779279005748	.0000E+00

H= 1.200 Y= .988918927519

YN=	.588921671224	.588921671224	.1110E-15
ZN=	1.011078370326	1.011078370326	.2220E-15
ZZ=	2.768975079665	2.768975079665	.0000E+00

H= 1.600 Y= .985252870604

YN=	.585255347482	.585255347482	.0000E+00
ZN=	1.014744690850	1.014744690850	.0000E+00
ZZ=	2.758709351343	-2.758709351343	.0000E+00

H= 2.000 Y= .981599946187

YN=	.581602616265	.581602616265	.0000E+00
ZN=	1.018397118862	1.018397118862	.0000E+00
ZZ=	2.748181688158	2.748181688158	.0000E+00

APPENDIX E

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 5

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SLTN1', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-10.0
      P=D(1)-D(2)
      V(1)=(-0.1-D(2))/P
      V(2)=(-10.0-D(2))/P
      V(3)=(-100.0-D(2))/P
      V(4)=(-1000.0-D(2))/P
      W(1)=(0.1+D(1))/P
      W(2)=(10.0+D(1))/P
      W(3)=(100.0+D(1))/P
      W(4)=(1000.0+D(1))/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.5
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 2 ON PROBLEM 5
      APN=(1.0-EXP(2.*Q))/Q+2.
      APD=1.0-EXP(2.*Q)
      A=APN/APD
      RPN=(-1.0+EXP(2.*Q))/Q-EXP(2.*Q)-1.0
      RPD=EXP(3.*Q)-1.5*EXP(2.*Q)+0.5
      R=RPN/RPD
      RQ=(1.0+(2.0-A)*Q)/(1.0-A*Q)
      WRITE(3,11)Q
      WRITE(3,10)A,R,RQ
      T=RQ**1.5
      YN1=R*Q*T+1.0+0.5*Q*(2.+R)
      YD=1.-0.5*Q*(2.0-3.*R)
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+1.0
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+1.0
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
```

```
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 2 , 6.7 ON PROBLEM 5',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'A R RQ',3(E15.6,2X)/)
   11 FORMAT(5X,' Q = ',2(F10.4,2X)/)
   40 FORMAT(5X,'H= ',F6.3,5X,'Y= ',E18.12,/)
   65 FORMAT(5X,'Y1= ',E18.12,5X,E18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2= ',E18.12,5X,E18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3= ',E18.12,5X,E18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4= ',E18.12,5X,E18.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT FOR ORDER 2 L=0.5 ON PROBLEM 5

V(I)= .1000E+01 .0000E+00 .9091E+01 .1000E+03

W(I)= .0000E+00 .1000E+01 .1009E+02 .1010E+03

Q = .0500

A R RQ .101666E+01 .459100E+00 .904837E+00

Q = 5.0000

A R RQ .180009E+01 .160033E+01 .453999E-04

YP(I)= .904837416688 .000045399930

H= 1.000 Y= .904837416688E+00

Y1=	.904837416688E+00	.904837416688E+00	.1110E-15
Y2=	.453999297625E-04	.453999297625E-04	.4065E-17
Y3=	.622833657183E+01	.622833657183E+01	.0000E+00
Y4=	.904791562895E+02	.904791562895E+02	.1421E-13

H= 2.000 Y= .818730750638E+00

Y1=	.818730750638E+00	.818730750638E+00	.1110E-15
Y2=	.206115362211E-08	.206115362211E-08	.3675E-21
Y3=	.744300680430E+01	.744300680430E+01	.8882E-15
Y4=	.818730748679E+02	.818730748679E+02	.1421E-13

H= 3.000 Y= .740818217370E+00

Y1=	.740818217370E+00	.740818217370E+00	.2220E-15
Y2=	.935762296884E-13	.935762296884E-13	.2175E-25
Y3=	.673171106801E+01	.673171106801E+01	.1776E-14
Y4=	.740818217481E+02	.740818217481E+02	.2842E-13

H= 4.000 Y= .670320042040E+00

Y1=	.670320042040E+00	.670320042040E+00	.2220E-15
Y2=	.424835425529E-17	.424835425529E-17	.1197E-29
Y3=	.609381856192E+01	.609381856192E+01	.1776E-14
Y4=	.670320012111E+02	.670320012111E+02	.1421E-13

```
H=   5.000      Y=   .606530655194E+00
Y1=   .606530655194E+00       .606530655194E+00       -.2220E-15
Y2=   .192874984796E-21       .192874984796E-21        .8504E-34
Y3=  -.551391504804E+01      -.551391504804E+01        .1776E-14
Y4=  -.606530655285E+02      -.606530655285E+02        .2132E-13
H=   6.000      Y=   .548811631187E+00
Y1=   .548811631187E+00       .548811631187E+00       -.3331E-15
Y2=   .875651076269E-26       .875651076270E-26        .4632E-38
Y3=  -.498919664791E+01      -.498919664791E+01        .3553E-14
Y4=  -.548811631270E+02      -.548811631270E+02        .2842E-13
H=   7.000      Y=   .496585298612E+00
Y1=   .496585298612E+00       .496585298612E+00       -.3331E-15
Y2=   .397544973591E-30       .397544973591E-30        .2453E-42
Y3=  -.451441180624E+01      -.451441180624E+01        .2665E-14
Y4=  -.496585298686E+02      -.496585298686E+02        .3553E-13
H=   8.000      Y=   .449328958761E+00
Y1=   .449328958761E+00       .449328958761E+00       -.3331E-15
Y2=   .180485138784E-34       .180485138785E-34        .1273E-46
Y3=  -.408480871662E+01      -.408480871662E+01        .2665E-14
Y4=  -.449328958828E+02      -.449328958828E+02        .3553E-13
H=   9.000      Y=   .406569654288E+00
Y1=   .406569654288E+00       .406569654288E+00       -.3331E-15
Y2=   .819401262398E-39       .819401262399E-39        .6501E-51
Y3=  -.369608776681E+01      -.369608776681E+01        .3109E-14
Y4=  -.406569654349E+02      -.406569654349E+02        .3553E-13
```

H 10.000 Y .367879135690E+00

Y1.	.367879135690E+00	.367879135690E+00	.3331E 15
Y2.	.372007597602E 43	.372007597602E 43	.3330E 55
Y3.	.334135850677E+01	.334135850677E+01	.3109E 14
Y4.	.367879435745E+02	.367879435745E+02	.3553E 13

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SLTN2', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-10.0
      P=D(1)-D(2)
      V(1)=(-0.1-D(2))/P
      V(2)=(-10.0-D(2))/P
      V(3)=(-100.0-D(2))/P
      V(4)=(-1000.0-D(2))/P
      W(1)=(0.1+D(1))/P
      W(2)=(10.0+D(1))/P
      W(3)=(100.0+D(1))/P
      W(4)=(1000.0+D(1))/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.5
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C ISTEP METHOD FOR THE CASE K=2 ORDER 4 ON PROBLEM 5
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,11)Q
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
    5 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YP(K)=YN1/YD
    1 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+1.0
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+1.0
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
```

```
ER4=EXY4(I)-Y4N(I)
WRITE(3,65)Y1N(I),EXY1(I),ER1
WRITE(3,70)Y2N(I),EXY2(I),ER2
WRITE(3,80)Y3N(I),EXY3(I),ER3
WRITE(3,75)Y4N(I),EXY4(I),ER4
CONTINUE
FORMAT(5X,'RESULT FOR ORDER 4 h=0.5 ON PROBLEM 5',/)
FORMAT(5X,'V(I)=',4(E12.4,2X)/)
FORMAT(5X,'W(I)=',4(E12.4,2X)/)
FORMAT(5X,'YP(I)=',2F18.12,2X/)
FORMAT(5X,'A R RQ',3(E15.6,2X)/)
FORMAT(5X,' Q = ',2(F10.4,2X)/)
FORMAT(5X,'H= ',F6.3,5X,'Y= ',E18.12,/)
FORMAT(5X,'Y1= ',E18.12,5X,E18.12,5X,E10.4,/)
FORMAT(5X,'Y2= ',E18.12,5X,E18.12,5X,E10.4,/)
FORMAT(5X,'Y3= ',E18.12,5X,E18.12,5X,E10.4,/)
FORMAT(5X,'Y4= ',E18.12,5X,E18.12,5X,E10.4,/)
STOP
END
```

RESULT FOR ORDER 4 h=0.5 ON PROBLEM 5

V(I)=	.1000E+01	.0000E+00	-.9091E+01	-.1000E+03
W(I)=	.0000E+00	.1000E+01	.1009E+02	.1010E+03
Q =	.0500			
A R RQ	.100333E+01	.120898E+00	.904837E+00	
Q =	5.0000			
A R RQ	.121662E+01	.315046E+00	.453999E-04	
YP(1)=	.904837416688	.000045399930		
H= 1.000	Y=	.904837416688E+00		
Y1=	.904837416688E+00	.904837416688E+00	.0000E+00	
Y2=	.453999297625E-04	.453999297625E-04	.1605E-16	
Y3=	-.822533657183E+01	-.822533657183E+01	.0000E+00	
Y4=	-.904791562895E+02	-.904791562895E+02	.0000E+00	
H= 2.000	Y=	.818730750638E+00		
Y1=	.818730750638E+00	.818730750638E+00	.0000E+00	
Y2=	.206115362244E-08	.206115362244E-08	.1457E-20	
Y3=	-.744300680430E+01	-.744300680430E+01	.0000E+00	
Y4=	-.818730748679E+02	-.818730748679E+02	.0000E+00	
H= 3.000	Y=	.740818217370E+00		
Y1=	.740818217370E+00	.740818217370E+00	.0000E+00	
Y2=	.935762296883E-13	.935762296884E-13	.9926E-25	
Y3=	-.673471106801E+01	-.673471106801E+01	.0000E+00	
Y4=	-.740818217481E+02	-.740818217481E+02	.0000E+00	
H= 4.000	Y=	.670320042040E+00		
Y1=	.670320042040E+00	.670320042040E+00	.0000E+00	
Y2=	.424835425529E-17	.424835425529E-17	.6008E-29	
Y3=	-.609381856492E+01	-.609381856492E+01	.0000E+00	
Y4=	-.670320042141E+02	-.670320042141E+02	.0000E+00	

```
H= 5.000      Y= .60653065194E+00
Y1= .606530655191E+00     .606530655191E+00     .1110E 15
Y2= .1928749817[illegible]E 21     .192874981796E 21     .3110E 33
Y3= .551391504804E+01     .551391504804E+01     .1776E 14
Y4= .606530655285E+02     .606530655285E+02     .1121E 13
H= 6.000      Y= .548811631187E+00
Y1= .548811631187E+00     .548811631187E+00     .1110E 15
Y2= .875651076268E 26     .875651076270E 26     .1858E 37
Y3= .49891966479[illegible]E+01     .498919664791E+01     .8882E 15
Y4= .548811631270E+02     .548811631270E+02     .1121E 13
H= 7.000      Y= .496585298612E+00
Y1= .496585298612E+00     .496585298612E+00     .1665E 15
Y2= [illegible]     .397511973591E 30     .9839E 42
Y3= [illegible]     .151411180621E+01     .1776E 14
Y4= [illegible]     .496585299686E+02     .1121E 13
H= 8.000      Y= .449328958761E+00
Y1= [illegible]     .449328956761E+00     .1665E 15
Y2= [illegible]     .180435138785E 31     .5105E 46
Y3= [illegible]     .408180871662E+01     .1776E 14
Y4= [illegible]     .449328958828E+02     .1121E 13
H= 9.000      Y= .406569651288E+00
Y1= [illegible]     .406569651288E+00     .1665E 15
Y2= [illegible]     .819401262399E 35     .2067E 50
Y3= [illegible]     .369608776681E+01     .1776E 14
Y4= [illegible]     .406569651349E+02     .1121E 13
   :
```

X= 10.000 Y= .367879435690E+00

Y1=	.367879435690E+00	.367879435690E+00	.1665E 15
Y2=	.372007597601E 43	.372007597602E 43	.1315E 54
Y3=	.334435850677E+01	.334435850677E+01	.1332E 14
Y4=	.367879435745E+02	.367879435745E+02	.1421E 13

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