International Journal of Mathematical Analysis, Vol. x, 2013, no. xx, xxx - xxx

Mappings with generalized weak contractive conditions and their fixed common fixed points in cone metric spaces

H.O. Olaoluwa

Department of Mathematics
University of Lagos
Akoka, Yaba, Lagos, Nigeria
Email: olu20_05@yahoo.fr

J.O. Olaleru

Department of Mathematics
University of Lagos
Akoka, Yaba, Lagos, Nigeria
Email: olaleru1@yahoo.co.uk

Abstract

We investigate and prove some fixed point results for weakly compatible mappings satisfying some generalized contractive condition in cone metric spaces and relate our findings to operators of integral type. The results are improvement and unification of many results in literature including Abbas, Rhoades and Nazir (2010), Bari and Vetro (2008) and Sabetghadam and Masiha (2010).

Subject Classification: 47H10

Keywords: Common fixed point, Cone metric spaces, Integral type, Weakly compatible pair.

1 Introduction

After Huang and Zhang [8] re-introduced cone metric spaces, several authors have extended many known fixed point results in usual metric spaces to cone metric spaces (e.g. [1,2,11-14,18]). Recently, F. Sabetghadam and H. P. Masiha [17] investigated the existence of the common fixed point of generalized φ-pairs in cone metric spaces, a notion previously studied by C. Di Bari and C. Vetro [6]. Our first aim in this manuscript is to make use of the simplest of such φ

[$\varphi(\omega) = k\omega$] to generalize and unify their results with those in [1]. The corollaries of our Theorems provide integral type conditions under which mappings in cone metric spaces have common fixed points. conditions which are more general than those in [12]. After several studies of integral type operators in metric spaces (e.g. [4,5,7,16,19]), the authors of [12] introduced the concept of integration in the setting of cone metric spaces and attempted to prove the existence of fixed point of a map satisfying the Biancari integral type condition (see [5]). However, in their paper, I.D. Arandelovic and D.J. Keckic [3] furnished a counterexample of the former theorem, suggesting by the way, an additional hypothesis in the proof thereof.
Here are some useful definitions and propositions stated in [8], [6] and [9].

Let E be a real Banach space. A subset $P \subset E$ is called a *cone* if:
(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in R$, $a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$ (interior of P).

A cone $P \subseteq E$ is called *normal* if there is $K > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. The cone is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq y$ for some $y \in E$, then there is a $x \in E$ such that $lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. In [15], it was shown that every regular cone is normal.

Let us assume that P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.1 [8] Let X be a nonempty set. Suppose that $d : X \times X \to P$ satisfies the following conditions:
(i) $\forall x, y \in X, d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$
Then d is called a *cone metric* on X, and (X, d) is called a *cone metric* space.

Example 1.2 [8] Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbf{R}$, $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Example 1.3 [13] Let $E = l^p$, $(1 \leq p < \infty)$, $P = \big\{\{x_n\}_{n\geq 1} \geq 0, \text{ for all n}\big\}$, $(X.\rho)$ a metric space and $d : X \times X \to E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Definition 1.4 [8] Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X.
(i) $\{x_n\}$ is said to be a *Cauchy* sequence if for every $c \in E$ with $0 \ll c$, there exists $N > 0$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$.
(ii) $\{x_n\}$ is said to be *convergent* to $x \in X$, denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$, if for every $c \in E$ with $0 \ll c$, there exists $N > 0$ such that for all $n \geq N$, $d(x_n, x) \ll c$.
It is shown in [6] that a convergent sequence in a cone metric (X, d) is a Cauchy sequence. When the converse is true, the cone metric space is said to be *complete*.

Proposition 1.5 [8] Let (X, d) be a cone metric space, P a normal cone and $\{x_n\}$, $\{y_n\}$ two sequences in X. Then:
(i) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(ii) $x_n \to x$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
(iii) The limit of $\{x_n\}$ is unique.
(iv) $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Definition 1.6 [9] The self mappings f and g of a cone metric space (X, d) are said to be *weakly compatible* if they commute at their coincidence points, that is, if $f(p) = g(p)$ for some $p \in X$, then $f(g(p)) = g(f(p))$.

The concept of weak compatibility is known to be the most general among all commutativity concepts in fixed point theory. For review of those notions of commutativity, see [9].

Definition 1.7 [17] Let P be a cone and let $\{\omega_n\}$ be a sequence in P. One says that $\omega_n \xrightarrow{\ll} 0$ if for every $\epsilon \in P$ with $0 \ll \epsilon$ there exists $N > 0$ such that $\omega_n \ll \epsilon$ for all $n \geq N$.

In the sequel, let $F : P \to P$, a non-decreasing mapping satisfying the following properties:
(F_1) For every $\omega_n \in P$, $\omega_n \xrightarrow{\ll} 0$ if and only if $F\omega_n \xrightarrow{\ll} 0$;
(F_2) For every $\omega_1, \omega_2 \in P$, $F(\omega_1 + \omega_2) \leq F(\omega_1) + F(\omega_2)$.

2 Generalized weak contractive conditions

We first state the following:

Proposition 2.1 *Let (X,d) be a cone metric space and let $A,B,S,T : X \to X$ be four mappings such that:*

$$\begin{aligned} F\Big(d(Sx,Ty)\Big) &\leq kF\Big(\psi\Big(d(Ax,By), d(Sx,Ax), d(Ty,By), \\ &\qquad d(Sx,By), d(Ty,Ax)\Big)\Big) \end{aligned}$$

where $0 < k < \frac{1}{2}$ and $\psi : P^5 \to P$ satisfies

$$\psi(a,a,b,0,c) \text{ and } \psi(a,b,a,c,0) \leq \begin{cases} 2a & \text{if } b \leq a \\ 2b & \text{if } a \leq b \\ a+b & \text{if } a-b \notin P \cup (-P) \end{cases} \tag{2.1}$$

$$\text{for all } c \leq a+b$$

and

$$\psi(t_1,t_2,t_3,t_4,t_5) \leq 2t \text{ when for all } i \in \{1,2,3,4,5\},\ t_i \leq t. \tag{2.2}$$

Suppose that A and S, B and T are weakly compatible with $S(X) \subset B(X)$, $T(X) \subset A(X)$ and such that one of $A(X), B(X), T(X), S(X)$ is a complete subspace of X. Then the maps A,B,S,T have a unique common fixed point.

Proof: Let $x_0 \in X$. We construct the following sequence:

$$\begin{cases} x_0 \in X \\ y_{2n+1} = Sx_{2n} = Bx_{2n+1} \\ y_{2n+2} = Tx_{2n+1} = Ax_{2n+2} \end{cases}$$

Let $d_n = d(y_n, y_{n+1})$.

$$\begin{aligned} F(d_{2n+1}) &= F(d(y_{2n+1}, y_{2n+2})) = F(d(Sx_{2n}, Tx_{2n+1})) \\ &\leq kF(\psi(d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \\ &\qquad d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n}))) \\ &\leq kF(\psi(d_{2n}, d_{2n}, d_{2n+1}, 0, d(y_{2n}, y_{2n+2}))). \end{aligned} \tag{2.3}$$

$d(y_{2n}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) = d_{2n} + d_{2n+1}$, hence,

$$\psi(d_{2n}, d_{2n}, d_{2n+1}, 0, d(y_{2n}, y_{2n+2})) \leq \begin{cases} 2d_{2n} & \text{if } \quad d_{2n+1} \leq d_{2n} \\ 2d_{2n+1} & \text{if } \quad d_{2n} \leq d_{2n+1} \\ d_{2n} + d_{2n+1} & \text{if } \quad d_{2n} \text{ and } d_{2n+1} \\ & \qquad \text{are not comparable.} \end{cases} \tag{2.4}$$

If $d_{2n} \leq d_{2n+1}$, then from (2.3) and (2.4), $F(d_{2n+1}) \leq kF(2d_{2n+1}) \leq 2kF(d_{2n+1}) < F(d_{2n+1})$, which is a contradiction.
If $d_{2n+1} \leq d_{2n}$, then from (2.3) and (2.4),

$$F(d_{2n+1}) \leq kF(2d_{2n}) \leq 2kF(d_{2n}). \tag{2.5}$$

If d_{2n} and d_{2n+1} are not comparable,

$$F(d_{2n+1}) \leq kF(d_{2n} + d_{2n+1}) \leq k[F(d_{2n}) + F(d_{2n+1})].$$

Thus

$$F(d_{2n+1}) \leq \frac{k}{1-k} F(d_{2n}). \tag{2.6}$$

Hence for all n, combining (2.5) and (2.6),

$$F(d_{2n+1}) \leq max\{2k, \frac{k}{1-k}\} F(d_{2n}) = 2kF(d_{2n}). \tag{2.7}$$

$$\begin{aligned} F(d_{2n}) &= F(d(y_{2n}, y_{2n+1})) = F(d(Sx_{2n}, Tx_{2n-1}) \\ &\leq kF(\psi(d(Ax_{2n}, Bx_{2n-1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n-1}), \\ &\quad d(Sx_{2n}, Bx_{2n-1}), d(Tx_{2n-1}, Ax_{2n}))) \\ &\leq kF(\psi(d_{2n-1}, d_{2n}, d_{2n-1}, d(y_{2n-1}, y_{2n+1}), 0)) \end{aligned}$$

$d(y_{2n-1}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) = d_{2n-1} + d_{2n}$, hence,

$$\psi(d_{2n-1}, d_{2n}, d_{2n-1}, d(y_{2n-1}, y_{2n+1}), 0) \leq \begin{cases} 2d_{2n-1} & \text{if} \quad d_{2n} \leq d_{2n-1} \\ 2d_{2n} & \text{if} \quad d_{2n-1} \leq d_{2n} \\ d_{2n-1} + d_{2n} & \text{if} \quad d_{2n-1} \text{ and } d_{2n} \\ & \quad\text{are not comparable.} \end{cases}$$

If $d_{2n-1} \leq d_{2n}$ then $F(d_{2n}) \leq kF(2d_{2n}) \leq 2kF(d_{2n}) < F(d_{2n})$, which is a contradiction.
If $d_{2n} \leq d_{2n-1}$,

$$F(d_{2n}) \leq kF(2d_{2n-1}) \leq 2kF(d_{2n-1}).$$

If d_{2n-1} and d_{2n} are not comparable,

$$F(d_{2n}) \leq kF(d_{2n-1} + d_{2n}) \leq k[F(d_{2n-1}) + F(d_{2n})].$$

Thus,

$$F(d_{2n}) \leq \frac{k}{1-k} F(d_{2n-1}).$$

Hence for all n,

$$F(d_{2n}) \leq max\{2k, \frac{k}{1-k}\} F(d_{2n-1}) = 2kF(d_{2n-1}). \tag{2.8}$$

Now, from (2.7) and (2.8) we have for all $n > 1$, $F(d_n) \leq hF(d_{n-1})$ where $h = 2k < 1$.

By induction $F(d_n) \leq hF(d_{n-1}) \leq h^2F(d_{n-2}) \leq ... \leq h^nF(d_0)$.
For $m > n$, we have:

$$\begin{aligned} F(d(y_n, y_m)) &\leq F(d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m)) \\ &\leq F(d_n) + F(d_{n+1}) + ... + F(d_{m-1}) \\ &\leq (h^n + h^{n+1} + ... + h^{m-n+1})F(d_0) \\ &\leq \tfrac{h^n}{1-h}F(d_0) \end{aligned}$$

As $n, m \to \infty$, $F(d(y_n, y_m)) \xrightarrow{\ll} 0$ hence by (F_2), $\{y_n\}$ is Cauchy.
Suppose that $A(X)$ is complete. As $\{y_{2n}\} \subset A(X)$, there exists $u \in X$ such that $y_n \to Au$. Let $v = Au$. Let us prove that $Su = v$.

$$\begin{aligned} F(d(y_{2n}, Su)) &= F(d(Su, Tx_{2n-1})) \\ &\leq kF(\psi(d(Au, Bx_{2n-1}), d(Su, Au), \\ &\quad d(Tx_{2n-1}), Bx_{2n-1}), d(Su, Bx_{2n-1}), d(Tx_{2n-1}, Au))) \\ &\leq kF(\psi(d(Au, y_{2n-1}), d(Su, Au), \\ &\quad d(y_{2n}, y_{2n-1}), d(Su, y_{2n-1}), d(y_{2n}, Au))). \end{aligned} \tag{2.9}$$

By the triangle inequality,

$$d(y_{2n-1}, y_{2n}) \leq d(y_{2n-1}, Au) + d(Au, y_{2n}),$$

$$d(Su, y_{2n-1}) \leq d(Su, Au) + d(Au, y_{2n-1}).$$

Since all the elements in the argument of ψ in (2.9) are less than $d(y_{2n-1}, Au)+ d(Au, y_{2n}) + d(Su, Au)$, by (2.2),

$$\begin{aligned} &\psi(d(Au, y_{2n-1}), d(Su, Au), d(y_{2n}, y_{2n-1}), d(Su, y_{2n-1}), d(y_{2n}, Au)) \\ &\qquad \leq 2d(y_{2n-1}, Au) + 2d(Au, y_{2n}) + 2d(Su, Au). \end{aligned}$$

Hence,

$$\begin{aligned} &kF(\psi(d(Au, y_{2n-1}), d(Su, Au), d(y_{2n}, y_{2n-1}), d(Su, y_{2n-1}), d(y_{2n}, Au))) \\ &\qquad \leq kF(2d(y_{2n-1}, Au) + 2d(Au, y_{2n}) + 2d(Su, Au)) \\ &\qquad \leq k(2F(d(y_{2n-1}, Au)) + 2F(d(Au, y_{2n})) + 2F(d(Su, Au))) \\ &\qquad \leq 2kF(d(y_{2n-1}, Au)) + 2kF(d(Au, y_{2n})) + 2kF(d(Su, Au)). \end{aligned} \tag{2.10}$$

Using the triangle inequality, (2.9) and (2.10), we have

$$\begin{aligned} F(d(Au, Su)) &\leq F(d(Au, y_{2n})) + F(d(y_{2n}, Su)) \\ &\leq F(d(Au, y_{2n})) + 2kF(d(y_{2n-1}, Au)) \\ &\quad +2kF(d(Au, y_{2n})) + 2kF(d(Su, Au)). \end{aligned}$$

Hence, $F(d(Su, Au)) \leq \frac{1+2k}{1+2k}F(d(Au, y_{2n})) + \frac{2k}{1-2k}F(d(y_{2n+1}, Au))$.
Since $d(Au, y_{2n}) \xrightarrow{\ll} 0$ and $d(Au, y_{2n-1}) \xrightarrow{\ll} 0$, then $F(d(Su, Au)) \xrightarrow{\ll} 0$ and

so $d(Su, Au) \xrightarrow{\ll} 0$. Thus $d(Su, Au) = 0$ i.e $v = Au = Su$.

Since $S(X) \subset B(X)$, there exists $w \in X$ such that $Su = Bw$.
Next we prove that $Bw = Tw$.

$$\begin{aligned} F(d(Bw, Tw)) &= F(d(Su, Tw)) \\ &\leq kF(\psi(d(Au, Bw), d(Su, Au), \\ &\quad d(Tw, Bw), d(Su, Bw), d(Tw, Au))) \\ &= kF(\psi(0, 0, d(Tw, Bw), 0, d(Tw, Bw))) \\ &\leq kF(2d(Tw, Bw)) \leq 2kF(d(Tw, Bw)) \end{aligned}$$

Thus, $(1-2k)F(d(Bw, Tw)) \leq 0$. Since $1-2k > 0$, $F(d(Bw, Tw)) = 0$ which implies that $d(Bw, Tw) = 0$ and so $v = Bw = Tw$.

Since $\{A, S\}$ and $\{B, T\}$ are weakly compatible, we have the following:

$$\begin{cases} Au = Su \Rightarrow ASu = SAu \Rightarrow Av = Sv, \\ Bw = Tw \Rightarrow BTw = TBw \Rightarrow Bv = Tv. \end{cases}$$

Let us prove that $Sv = Tv$.

$$\begin{aligned} F(d(Sv, Tv)) &\leq kF(\psi(d(Av, Bv), d(Sv, Av), d(Tv, Bv), d(Sv, Bv), d(Tv, Av))) \\ &= kF(\psi(d(Sv, Tv), 0, 0, d(Sv, Tv), d(Sv, Tv))) \\ &\leq kF(2d(Sv, Tv)) \leq 2kF(d(Sv, Tv)) \end{aligned}$$

Thus $(1-2k)F(d(Sv, Tv)) \leq 0$, hence, $F(d(Sv, Tv)) = 0$, and so $d(Sv, Tv) = 0$ i.e $Sv = Tv$. We have $Av = Sv = Bv = Tv$. Now let us prove that $v = Sv$.

$$\begin{aligned} F(d(y_{2n+1}, Tv)) &= F(d(Sx_{2n}, Tv) \\ &\leq kF(\psi(d(Ax_{2n}, Bv), d(Sx_{2n}, Ax_{2n}), \\ &\quad d(Tv, Bv), d(Sx_{2n}, Bv), d(Tv, Ax_{2n}))) \\ &= kF(\psi(d(y_{2n}, Tv), d(y_{2n+1}, y_{2n}), 0, \\ &\quad d(y_{2n+1}, Tv), d(Tv, y_{2n}))). \end{aligned} \tag{2.11}$$

Since $d(y_{2n}, Tv) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, Tv)$, we have from (2.2),

$$\begin{aligned} &\psi(d(y_{2n}, Tv), d(y_{2n+1}, y_{2n}), 0, d(y_{2n+1}, Tv), d(Tv, y_{2n})) \\ &\qquad \leq 2d(y_{2n}, y_{2n+1}) + 2d(y_{2n+1}, Tv). \end{aligned}$$

Hence from (2.11),

$$\begin{aligned} F(d(y_{2n+1}, Tv)) &\leq k(2F(d(y_{2n}, y_{2n+1})) + 2F(d(y_{2n+1}, Tv))) \\ &= 2kF((d(y_{2n}, y_{2n+1})) + 2kF(d(y_{2n+1}, Tv)). \end{aligned}$$

which yields $F(d(y_{2n+1}, Tv)) \leq \frac{2k}{1-2k} F(d(y_{2n}, y_{2n+1}))$.
Since $d(y_{2n}, y_{2n+1}) \xrightarrow{\ll} 0$, then $F(d(y_{2n+1}, Tv)) \xrightarrow{\ll} 0$ and $d(y_{2n+1}, Tv) \xrightarrow{\ll} 0$.
So $y_{2n+1} \to Tv$ i.e $v = Tv$ from the uniqueness of limit of $\{y_n\}$.

So A, B, S, T have a common fixed point. The uniqueness follows from the contractive condition. The same result holds if we suppose that one of $S(X), T(X)$, $B(X)$ is complete.

Remark. If $S = T$, $A = B$ and $\psi(t_1, t_2, t_3, t_4, t_5) = t_2 + t_3$ (respectively $\psi(t_1, t_2, t_3, t_4, t_5) = t_4 + t_5$) in Proposition 2.1, we obtain Theorem 2.9 (respectively Theorem 2.12) in [17] and the corollaries ([2]).

Theorem 2.2 *Let (X, d) be a cone metric space and let $A, B, S, T : X \to X$ be four mappings such that:*

$$F(d(Sx,Ty)) \leq kF\Big(\psi\Big(d(Ax,By), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ty,Ax)\Big)\Big)$$

where F in addition is such that for every $\omega \in P$, $F(\omega) = 2F\Big(\frac{\omega}{2}\Big)$, $0 < k < 1$ and $\psi : P^5 \to P$ satisfies

$$\psi(a,a,b,0,c) \text{ and } \psi(a,b,a,c,0) \leq \begin{cases} a & \text{if } b \leq a \\ b & \text{if } a \leq b \\ \frac{a+b}{2} & \text{if } a-b \notin P \cup (-P) \end{cases} \quad (2.12)$$

$$\text{for all } c \leq a+b$$

and

$$\psi(t_1,t_2,t_3,t_4,t_5) \leq t \text{ when for all } i \in \{1,2,3,4,5\},\ t_i \leq t. \quad (2.13)$$

Suppose that $\{A, S\}$ and $\{B, T\}$ are weakly compatible with $S(X) \subset B(X)$, $T(X) \subset A(X)$ and such that one of $A(X), B(X), T(X), S(X)$ is a complete subspace of X. Then the self-mappings A, B, S, T have a unique common fixed point.

Proof: Using the additional property of F, i.e, $\forall \omega \in P$, $F(\omega) = 2F\Big(\frac{\omega}{2}\Big)$, the above contractive condition can be written thus:

$$F(d(Sx,Ty)) \leq \frac{k}{2}F(\psi'(d(Ax,By), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ty,Ax)))$$

where $\psi' = 2\psi$ satisfies the hypothesis in Proposition 2.1 and $0 < \frac{k}{2} < \frac{1}{2}$. ■

Letting $F = Id$ and $\psi(t_1, t_2, t_3, t_4, t_5) \in \Big\{t_1, t_2, t_3, \frac{t_4+t_5}{2}\Big\}$ (respectively $\psi(t_1, t_2, t_3, t_4, t_5) \in \Big\{t_1, \frac{t_2+t_3}{2}, \frac{t_4+t_5}{2}\Big\}$) in the contractive condition of Theorem 2.2, we obtain the following corollaries:

Corollary 2.3 [1] Let A, B, S and T be self mappings of a cone metric space$X$ with cone P having a non-empty interior, satisfying $S(X) \subset B(X)$, $T(X) \subset A(X)$ and

$$d(Sx, Ty) \leq hu$$

where $h \in (0, 1)$ and

$$u \in \left\{ d(Ax, By), d(Sx, Ax), d(Ty, By), \frac{d(Sx, By) + d(Ty, Ax)}{2} \right\}$$

for all $x, y \in X$. If $\{A, T\}$ and $\{B, S\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Corollary 2.4 *Let A, B, S and T be self mappings of a cone metric space$X$ with cone P having a non-empty interior, satisfying $S(X) \subset B(X)$, $T(X) \subset A(X)$ and*

$$d(Sx, Ty) \leq hu$$

where $h \in (0, 1)$ and

$$u \in \left\{ d(Ax, By), \frac{d(Sx, Ax) + d(Ty, By)}{2}, \frac{d(Sx, By) + d(Ty, Ax)}{2} \right\}$$

for all $x, y \in X$. If $\{A, T\}$ and $\{B, S\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Remark. Corollary 2.3 is Theorem 2.2 in [1]. Also, when (X, d) is a metric space, the contractive condition in our main result is the same as in Theorem 2.3 in [6].

3 Some general conditions of integral type

We start with some definitions, examples and properties as stated in [12].

Definition 3.1 [12] Suppose that P is a normal cone in E. Let $a, b \in E$ and $a < b$. We define:

$$[a, b] := \{x \in E : x = tb + (1 - t)a, \text{ where } t \in [0, 1]\}$$

$$[a, b) := \{x \in E : x = tb + (1 - t)a, \text{ where } t \in [0, 1)\}$$

Definition 3.2 [12] The set $\{a = x_0, x_1, ..., x_n = b\}$ is called a partition for $[a, b]$ if and only if the sets $[x_{i-1}, x_i]$, $1 \leq i \leq n$, are pairwise disjoint and

$$[a, b] = \left\{ \bigcup_{i=1}^{n} [x_{i-1}, x_i) \right\} \cup \{b\}$$

Definition 3.3 [12] Suppose that P is a normal cone in E, $\phi : [a, b] \to P$ a map. ϕ is said to be *integrable on* $[a, b]$ *with respect to cone* P (or *cone integrable function*) iff for all partition Q of $[a, b]$

$$lim_{n\to\infty} L_n^{Con}(\phi, Q) = S^{Con} = lim_{n\to\infty} U_n^{Con}(\phi, Q)$$

where S^{Con} must be unique and:

$$\begin{cases} L_n^{Con} = \sum_{i=0}^{n-1} \phi(x_i)\|x_i - x_{i+1}\| & \text{(Cone lower summation)} \\ U_n^{Con} = \sum_{i=0}^{n-1} \phi(x_{i+1})\|x_i - x_{i+1}\| & \text{(Cone upper summation).} \end{cases}$$

We note

$$S^{Con} = \int_a^b \phi(x) d_P(x) = \int_a^b \phi d_P$$

The set of all cone integrable functions $\phi : [a, b] \to P$ is denoted $L^1([a, b], P)$.

Definition 3.4 [12] The function $\phi : P \to E$ is called *subadditive cone integrable function* iff $\forall a, b \in P$

$$\int_0^{a+b} \phi d_P \leq \int_0^a \phi d_P + \int_0^b \phi d_P$$

Example 3.5 [12] Let $E = X = \mathbf{R}$, $d(x, y) = |x - y|$, $P = [0, +\infty)$ and $\phi(t) = \frac{1}{t+1}$ $\forall t > 0$. Then ϕ is a subbaditive cone integral function.

We are now in position to state the following proposition

Proposition 3.6 *Let* (X, d) *be a cone metric space and let and* P *a normal cone. Let* $\phi : P \to P$ *be a nonvanishing map and a subbaditive cone integrable on each* $[a, b]$*. Let* $A, B, S, T : X \to X$ *be four mappings such that:*

$$\int_0^{d(Sx,Ty)} \phi(t) d_P(t) \leq k \int_0^{M(x,y)} \phi(t) d_P(t), \quad k \in [0, \frac{1}{2}) \tag{3.1}$$

where

$$M(x, y) = \psi\Big(d(Ax, By), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\Big), \tag{3.2}$$

and $\psi : P^5 \to P$ *is a mapping satisfying* (2.1) *and* (2.2) *in Proposition 2.1. Suppose also that the function of* $y \to \int_0^y \phi d_P$ *is invertible and that the inverse is continuous in 0. If* $\{A, S\}$ *and* $\{B, T\}$ *are pairs of weakly compatible mappings with* $S(X) \subset B(X)$*,* $T(X) \subset A(X)$ *and such that one of* $A(X), B(X), T(X), S(X)$ *is a complete subspace of* X*, then the self-mappings* A, B, S, T *have a unique common fixed point.*

Proof. Proposition 3.6 is a corollary of Proposition 2.1 when $F(y) = \int_0^y \phi d_P$. Under this case, F satisfies conditions (F_1) and (F_2). (F_2) results from the subbaditivity of ϕ. The condition (F_1) results from the continuity of F and its inverse in 0. In fact, in a normal cone, if $\omega_n \xrightarrow{\ll} \omega$, then ω_n converges to ω. Now, since F is continuous in 0, for every sequence ω_n converging to 0, $F(\omega_n)$ converges to $F(0) = 0$. Since F^{-1} is continuous, given any sequence $F(\omega_n)$ converging to 0, $F^{-1}(F(\omega_n)) = \omega_n$ converges to $F^{-1}(0) = 0$; thus (F_1) is satisfied.

Theorem 3.7 *Let (X, d) be a cone metric space and let and P a normal cone. Let $\phi : P \to P$ be a nonvanishing map, a subbaditive cone integrable on each $[a, b]$ and such that $\int_0^a \phi d_P = 2\int_0^{\frac{a}{2}} \phi d_P$. Let $A, B, S, T : X \to X$ be four mappings such that:*

$$\int_0^{d(Sx,Ty)} \phi(t) d_P(t) \leq k \int_0^{M(x,y)} \phi(t) d_P(t), \quad k \in [0, 1) \tag{3.3}$$

where

$$M(x, y) = \psi\Big(d(Ax, By), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\Big), \tag{3.4}$$

and $\psi : P^5 \to P$ is a mapping satisfying (2.12) *and* (2.13) *in Theorem 2.2. Suppose also that the function of $y \to \int_0^y \phi d_P$ is invertible and that F and F^{-1} are continuous in 0. If $\{A, S\}$ and $\{B, T\}$ are pairs of weakly compatible mappings with $S(X) \subset B(X)$, $T(X) \subset A(X)$ and such that one of $A(X), B(X), T(X), S(X)$ is a complete subspace of X, then the self-mappings A, B, S, T have a unique common fixed point.*

Proof. Theorem 3.7 is a corollary of Theorem 2.2 when $F(y) = \int_0^y \phi d_P$.

Remark. Theorem 3.7 is the cone version of Theorem 2.4 in [6], extending thus to cone metric spaces many other known results in metric spaces. It also furnishes the additional hypothesis to those in Theorem 2.9 of [12] which would make it valid, as suggested in [3].
The following corollary is a result of existence of the fixed point of a single map

Corollary 3.8 *Let (X, d) be a complete cone metric space and let and P a normal cone. Let $\phi : P \to P$ be a nonvanishing map, a subbaditive cone integrable on each $[a, b]$ and such that $\int_0^a \phi d_P = 2\int_0^{\frac{a}{2}} \phi d_P$. Let $S : X \to X$ be a mapping such that:*

$$\int_0^{d(Sx,Sy)} \phi(t) d_P(t) \leq k \int_0^{M(x,y)} \phi(t) d_P(t), \quad k \in [0, 1) \tag{3.5}$$

where

$$M(x,y)=\psi\bigg(d(x,y),d(Sx,x),d(Sy,y),d(Sx,y),d(Sy,x)\bigg), \tag{3.6}$$

and $\psi : P^5 \to P$ *is a mapping satisfying* (2.12) *and* (2.13) *in Theorem 2.2. Suppose also that the function of* $y \to \int_0^y \phi d_P$ *is invertible and that the inverse is continuous in 0. Then* S *has a unique common fixed point.*

Proof: Take $A = B = Id_P$ and $S = T$ in Theorem 3.7.

References

[1] M. Abbas, B.E. Rhoades and T. Nazir, Common fixed points for four maps in cone metric spaces, *Applied Mathematics and Computation,* **216** (2010) 80-86.

[2] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *Journal of Mathematical Analysis and Applications,* **341 (1)** (2008) 416420.

[3] I. D. Arandelovic and D. J. Kecki, A Counterexample on a Theorem by Khojasteh, Goodarzi, and Razani, *Fixed Point Theory and Applications,* vol 2010, Article ID 470141, 6 pages, 2010.

[4] I. Atun, D. Turkoglu and B.E. Rhoades, Fixed points of weakly compatible maps satisfying a general contractive of integral type, *Fixed Point Theory and Applications,* vol. 2007, Article ID 17301, 9 pages, 2007.

[5] A. Biancari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *International Journal of Mathematics and Mathematical Sciences,* **29 (9)** (2002) 531-536.

[6] C. Di Bari and C. Vetro, ϕ-pairs and common fixed points in cone metric spaces, *Rendiconti del Circolo Matematico di Palermo,* **57 (2)** (2008) 279285.

[7] A.Djoudi and F. Merghadi, Common fixed point theorems for maps under a contractive condition of integral type, *Journal of Mathematical Analysis and Applications,* **341** (2008) 953-960.

[8] L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications,* **332 (2)** (2007)1468-1476.

[9] G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, *Far East J. Math. Sci.,* **4** (1996) 199-215.

[10] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, Common Fixed Point Theorems for Weakly Compatible Pairs on Cone Metric Spaces, *Fixed Point Theory and Applications,* vol. 2009, Article ID 643840, 13 pages, 2009.

[11] F. Khojasteh, Z. Goodarzi and A. Razani, Erratum to "Some fixed point theorems of integral type contraction in cone metric spaces," *Fixed Point Theory and Applications,* vol. 2011, Article ID 346059, 2 pages, 2011.

[12] F. Khojasteh, Z. Goodarzi and A. Razani, Some fixed point theorems of integral type contraction in cone metric spaces, *Fixed Point Theory and Applications,* vol. 2010, Article ID 189684, 13 pages, 2010.

[13] J.O. Olaleru, Some Generalizations of Fixed Point Theorems in Cone Metric Spaces, *Fixed Point Theory and Applications*, vol 2009, Article ID 657914, 10 pages, 2009.

[14] S. Radenovich and B.E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, *Computers and Mathematics with Applications,* **57** (2009) 1701-1707.

[15] S. Rezapour and R. Hamlbarani, Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings", *Journal of Mathematical Analysis and Applications,* **345 (2)** (2008) 719724.

[16] B.E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, *International Journal of Mathematics and Mathematical Sciences,* **3** (2003) 4007-4013.

[17] F. Sabetghadam and H.P. Masiha, Common fixed points for generalized ϕ-pair mappings on cone metric spaces, *Fixed Point Theory and Applications*, vol. 2010, Article ID 718340, 8 pages, 2010.

[18] G. Song, X. Sun, Y. Zhao and G. Wang, New common fixed point theorems for maps on cone metric spaces, *Applied Mathematics Letters,* **23** (2010) 1033-1037.

[19] P. Vijayaraju, B.E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *International Journal of Mathematics and Mathematical Sciences,* **15** (2005) 2359-2364.

Received: February 22, 2013