Convergence Theorems on Generalized Strongly Successively Φ-pseudocontractive Mappings in the Intermediate Sense

G. A. Okeke[1*] and H. Olaoluwa[1]

[1] *Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria*

Research Article

Abstract

We introduce a new class of nonlinear mappings, the class of generalized strongly successively Φ-pseudocontractive mappings in the intermediate sense and prove the convergence of Mann type iterative scheme to their fixed points. Our results improves and generalizes several other results in literature.

1 Introduction

Let E be an arbitrary real normed linear space with dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing.

In the sequel, we give the following definitions which will be useful in this study

Definition 1.1. Let C be a nonempty subset of real normed linear space E. A mapping $T : C \to E$ is said to be:

(1) *strongly pseudocontractive* (Kim *et al.* [13]) if for all $x, y \in C$, there exists a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \tag{1.2}$$

(2) *φ-strongly pseudocontractive* (Kim *et al.* [13]) if for all $x, y \in C$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|, \tag{1.3}$$

Corresponding author: E-mail: gaokeke1@yahoo.co.uk

The class of ϕ-strongly pseudocontractive mappings includes the class of strongly pseudocontractive mappings by setting $\phi(s) = ks$ for all $s \in [0, \infty)$. However, the converse is not true.

(3) *generalized Φ-pseudocontractive* (Alber *et al.* [1], Chidume and Chidume [7]) if for all $x, y \in C$, there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|), \tag{1.4}$$

(4) *asymptotically generalized Φ-pseudocontractive* (Kim *et al.* [13]) with sequence $\{k_n\}$ if for each $n \in \mathbb{N}$ and $x, y \in C$, there exists a constant $k_n \geq 1$ with $\lim_{n \to \infty} k_n = 1$, a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 - \Phi(\|x - y\|), \tag{1.5}$$

The class of asymptotically generalized Φ-pseudocontractive was introduced by Kim *et al.* [13].

(5) *strongly successively pseudo-contractive* (Huang [11]) if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ and a constant $k \in (0, 1)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq (1 - k)\|x - y\|^2, \quad \forall \ n \in \mathbb{N}; \tag{1.6}$$

(6) *strongly successively ϕ-pseudocontractive* (Huang [11]) if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|; \tag{1.7}$$

(7) *generalized strongly successively Φ-pseudocontractive* (Huang [11]) if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ and a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|) \tag{1.8}$$

The map $T : C \to C$ is said to be *asymptotically nonexpansive mappings in the intermediate sense* (Bruck *et al.* [2]) if it is continuous and the following inequality holds:

$$\limsup_{n \to \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.9}$$

Observe that if we define

$$\xi_n = \max \left\{ 0, \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \right\}, \tag{1.10}$$

then $\xi_n \to 0$ as $n \to \infty$. It follows that (1.9) is reduced to

$$\|T^n x - T^n y\| \leq \|x - y\| + \xi_n, \quad \forall n \geq 1, \quad \forall x, y \in C. \tag{1.11}$$

In 1993, Bruck *et al.* [2] introduced the class of *asymptotically nonexpansive mappings in the intermediate sense.*
Sahu *et al.* [24] introduced the class of *asymptotically strict pseudocontractive mappings in the intermediate sense* as follows

Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ will be called an asymptotically k-strict pseudocontractive mappings in the intermediate sense with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} \gamma_n = 0$ such that

$$\limsup_{n \to \infty} \sup_{x, y \in C} \left(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - T^n x - (y - T^n y)\|^2 \right) \leq 0. \tag{1.12}$$

Assume that

$$c_n := \max \left\{ 0, \sup_{x, y \in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \right\}. \tag{1.13}$$

Then $c_n \geq 0$ for all $n \in \mathbb{N}$, $c_n \to 0$ as $n \to \infty$ and (1.12) reduces to the relation

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n \tag{1.14}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Qin *et al.* [21] introduced the class of asymptotically pseudocontractive mappings in the intermediate sense.

Definition 1.2. (Qin *et al.* [21]) Let C be a nonempty closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n \to \infty} \sup_{x,y \in C} \left(\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2 \right) \leq 0, \tag{1.15}$$

where $\{k_n\}$ is a sequence in $[1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Put

$$\nu_n = \max \left\{ 0, \sup_{x,y \in C} \left(\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2 \right) \right\}. \tag{1.16}$$

It follows that $\nu_n \to 0$ as $n \to \infty$. Then, (1.15) is reduced to the following:

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 + \nu_n, \quad \forall n \geq 1, \ x, y \in C. \tag{1.17}$$

They proved weak convergence theorems for this class of nonlinear mappings. They also established some strong convergence results without any compact assumption by considering the hybrid projection methods. Olaleru and Okeke [19] proved a strong convergence of Noor type scheme for a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness.

Motivated by the above facts, we now introduce the following class of nonlinear mappings

Definition 1.3. Let C be a nonempty subset of real normed linear space E. A mapping $T : C \to C$ is said to be *generalized strongly successively Φ-pseudocontractive mapping in the intermediate sense* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(x - y) \in J(x - y)$ satisfying

$$\limsup_{n \to \infty} \sup_{x,y \in C} \left(\langle T^n x - T^n y, j(x - y) \rangle - \|x - y\|^2 + \Phi(\|x - y\|) \right) \leq 0. \tag{1.18}$$

Put

$$\tau_n = \max \left\{ 0, \sup_{x,y \in C} \left(\langle T^n x - T^n y, j(x - y) \rangle - \|x - y\|^2 + \Phi(\|x - y\|) \right) \right\}. \tag{1.19}$$

It follows that $\tau_n \to 0$ as $n \to \infty$. Hence (1.18) is reduced to the following

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 + \tau_n - \Phi(\|x - y\|) \quad \forall n \in \mathbb{N}, \ x, y \in C. \tag{1.20}$$

We remark that if $\tau_n = 0$ for all $n \in \mathbb{N}$, the class of generalized strongly successively Φ-pseudocontractive in the intermediate sense is reduced to the class of generalized strongly successively Φ-pseudocontractive maps.

Example 1.4. Let $E = (-\infty, \infty)$ with the usual norm, $\Phi(t) = \frac{t^2}{3}$ for each $t \in [0, \infty)$ and $a_n = 2^{-n}$ for $n \geq 0$. Take $C = [0, 1] \cup \{2\}$

$$Tx = \begin{cases} 0 & \text{if } x \in \{0, 4\}, \\ 4 & \text{if } x = 1, \\ a_n - x, & \text{if } x \in [\frac{1}{2}(a_{n+1} + a_n), a_n), \\ x - a_{n+1}, & \text{if } x \in [a_{n+1}, \frac{1}{2}(a_{n+1} + a_n)). \end{cases}$$

for each $n \geq 0$, observe that $F(T) := \{x \in C : Tx = x\} = \{0\}$ and T is not continuous at

$x = 1$. It is easy to see that T is generalized strongly successively Φ-pseudocontractive mapping in the intermediate sense with sequence $\tau_n = \frac{1}{n^2}$.

Let C be a nonempty of a normed linear space E. A mapping $T : C \to E$ is said to be *Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \tag{1.21}$$

for all $x, y \in C$ and *generalized Lipschitzian* (Kim *et al.* [13]) if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1) \tag{1.22}$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *uniformly L-Lipschitzian* (Kim *et al.* [13]) if for each $n \in \mathbb{N}$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.23}$$

for all $x, y \in C$.

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. Every mapping with a bounded range is a generalized Lipschitzian mapping. The following example shows that the class of generalized Lipschitzian mappings properly contains the class of Lipschitzian mappings and that of mappings with bounded range.

Example 1.5. (Chang *et al.* [5]). Let $E = (-\infty, \infty)$ and $T : E \to E$ be defined by

$$Tx = \begin{cases} x - 1 & \text{if } x \in (-\infty, -1), \\ x - \sqrt{1 - (x + 1)^2} & \text{if } x \in [-1, 0), \\ x + \sqrt{1 - (x - 1)^2} & \text{if } x \in [0, 1], \\ x + 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

Sahu [22] introduced a new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings.

Definition 1.6. (Sahu [22]). Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$.

(1) A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \tag{1.24}$$

for all $x, y \in C$.

The infimum of constants k_n in (1.24) is called *nearly Lipschitz constant* and is denoted by $\eta(T^n)$.

(2) A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be *nearly uniformly L-Lipschitzian* if $k_n = L$ for all $n \in \mathbb{N}$, i.e.

$$\|T^n x - T^n y\| \leq L(\|x - y\| + a_n) \tag{1.25}$$

and *nearly asymptotically nonexpansive* if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n \to \infty} k_n = 1$.

It has been shown by Sahu [22] that the class of nearly uniformly L-Lipschitzian is not necessarily continuous. Sahu [22] extended the results of Goebel and Kirk [9] to demicontinuous mappings and proved that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every demicontinuous nearly asymptotically nonexpansive self-mapping of C has a fixed point.

Kim *et al.* [13] prove the following results

Theorem KSN. (Kim *et al.* [13]). Let C be a nonempty convex subset of a real Banach space E and $T : C \to C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and asymptotically generalized Φ-hemicontractive mapping with sequence $\{k_n\}$ and $F(T) := \{x \in C : x = Tx\} \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the conditions:

(i) $\{\frac{a_n}{\alpha_n}\}$ is bounded,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

Let $\{x_n\}$ be the sequence in C generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \in \mathbb{N}. \tag{1.26}$$

Then the sequence $\{x_n\}$ in C defined by (1.26) converges strongly to a unique fixed point of T.

The natural question will be as follows.

Problem 1.7. Is it possible to extend Theorem KSN to the class of generalized strongly successively Φ-pseudocontractive mappings in the intermediate sense introduced in this study.

We define the modified Mann iteration with errors as follows:

$$u_{n+1} = (1 - \alpha_n - \gamma_n)u_n + \alpha_n T^n u_n + \gamma_n \xi_n, \tag{1.27}$$

and the modified Ishikawa iteration with errors by

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n \nu_n, \\ \\ y_n = (1 - \alpha_n' - \gamma_n')x_n + \alpha_n' T^n x_n + \gamma_n' \omega_n, \end{cases} \tag{1.28}$$

where the sequences $\{\alpha_n\}, \{\alpha_n'\}, \{\gamma_n\}, \{\gamma_n'\} \subseteq [0, 1]$ satisfy

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \alpha_n' = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \ \gamma_n = o(\alpha_n), \ \lim_{n \to \infty} \gamma_n' = 0, \tag{1.29}$$

and the sequences $\{\xi_n\}, \{\nu_n\}, \{\omega_n\}$ are bounded.

Huang [11] obtained the following convergence results for the class of generalized strongly successively Φ-pseudocontractive mappings.

Theorem H. (Huang [11]). Let E be a real uniformly smooth space and let $T : E \to E$ be a generalized strongly successively Φ-pseudocontractive mapping with bounded range. The sequences $\{u_n\}$ and $\{x_n\}$ are defined by (1.28) and (1.29) respectively, with $\{\alpha_n\}, \{\alpha_n'\}, \{\gamma_n\}, \{\gamma_n'\} \subseteq [0, 1]$ satisfying (1.29) and $\{\xi_n\}, \{\nu_n\}, \{\omega_n\}$ being bounded. Then for $u_1, x_1 \in E$, the following two assertions are equivalent:

(i) modified Mann iteration with errors (1.27) converges to the fixed point $x^* \in F(T)$;

(ii) modified Ishikawa iteration with errors (1.28) converges to the fixed point $x^* \in F(T)$.

Mann [15] introduced the Mann iterative scheme and used it to prove the convergence of the sequence to the fixed points for which the Banach principle is not applicable. Ishikawa [12], introduced an iterative process to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann iterative scheme failed to converge.

It is our purpose in this study to use the concept of nearly uniformly L-Lipschitzian (not necessarily continuous) mappings to prove a strong convergence result for the class of *generalized strongly successively Φ-pseudocontractive mappings in the intermediate sense* in a general Banach space. Our results is an improvement of several other results in literature.

The following Lemmas will be useful in this study

Lemma 1.7. (Chang [4]). Let E be a Banach space. Then for each $x, y \in E$, there exists $j(x + y) \in J(x + y)$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 1.8. (Osilike and Aniagborsor [20]). Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that

$$\delta_{n+1} \le (1 + \beta_n)\delta_n + \gamma_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 1.9. (Moore and Nnoli [16]). Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ a real sequence in $[0,1]$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n$$

for all $n \ge n_0$, where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o(\lambda_n)$, then $\lim_{n\to\infty} \theta_n = 0$.

Lemma 1.10. (Kim *et al.* [13]). Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\sigma_n\}$ be four sequences of nonnegative numbers such that

$$\delta_{n+1}^2 \le (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n\to\infty} \delta_n$ exists.

2 Main Results

We prove the following Lemma which will be needed in this study.

Lemma 2.1. Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{\rho_n\}$ be five sequences of nonnegative numbers such that

$$\delta_{n+1}^2 \le (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \tag{2.1}$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \rho_n < \infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n\to\infty} \delta_n$ exists.

Proof. Using (2.1), we obtain

$$
\begin{aligned}
\delta_{n+1}^2 &\le (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \\
&\le (1 + \beta_n)\delta_n^2 + 2\gamma_n(\delta_n^2 + \sigma_n^2) + \rho_n^2 \\
&\le (1 + \beta_n + 2\gamma_n)\delta_n^2 + 2\gamma_n\sigma_n^2 + \rho_n^2.
\end{aligned}
\tag{2.2}
$$

Since $\{\sigma_n\}$ is bounded and $\sum_{n=1}^{\infty} \rho_n < \infty$, then by Lemma 1.8, it follows that $\lim_{n\to\infty} \delta_n$ exists. The proof of Lemma 2.1 is completed. $\square$

Theorem 2.2. Let C be a nonempty convex subset of a real Banach space E and $T : C \to C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and *generalized strongly successively Φ-pseudocontractive mapping in the intermediate sense* with sequence $\{\tau_n\}$ as defined in (1.20) and $F(T) := \{x \in C : Tx = x\} \ne \emptyset$. Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \in \mathbb{N}, \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in [0,1]. Assume that the following conditions are satisfied:
   (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$.
Then the sequence $\{x_n\}$ in C defined by (2.3) converges strongly to a fixed point of T.

Proof. Fix $p \in F(T)$, using (1.20) and (2.3) and set

$A_n := 2\alpha_n^2 L(1+L)$

and

$B_n := 1 - 2\alpha_n - 2\alpha_n^2 L(1+L).$

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \alpha_n \|T^n x_n - x_n\| \\
&\leq \alpha_n (\|T^n x_n - p\| + \|x_n - p\|) \\
&\leq \alpha_n (L\|x_n - p\| + a_n) + \|x_n - p\|) \\
&\leq \alpha_n (1+L)\|x_n - p\| + a_n L.
\end{aligned}
\tag{2.4}
$$

Using (1.20), (1.25), (2.3), (2.4) and Lemma 1.7, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T^n x_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\quad + \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p)\rangle\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + \|T^n x_n - T^n x_{n+1}\| \times \|x_{n+1} - p\|\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + L(\|x_{n+1} - x_n\| + a_n)\|x_{n+1} - p\|\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + L(\alpha_n(1+L)\|x_n - p\| + a_n L + a_n)\|x_{n+1} - p\|\}
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + \alpha_n L(1+L)\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)\|x_{n+1} - p\|\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|)\} \\
&\quad + 2\alpha_n^2 L(1+L)\left\{\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2 + \|x_{n+1} - p\|^2\right\}.
\end{aligned}
\tag{2.5}
$$

From (2.5), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(\frac{(1-\alpha_n)^2}{1-2\alpha_n-2\alpha_n^2 L(1+L)}\right)\|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{1-2\alpha_n-2\alpha_n^2 L(1+L)} \\
&\quad - \frac{2\alpha_n \Phi(\|x_{n+1}-p\|)}{1-2\alpha_n-2\alpha_n^2 L(1+L)} + \frac{\alpha_n^2 L(1+L)}{1-2\alpha_n-2\alpha_n^2 L(1+L)}\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2 \\
&= \left(\frac{(1-\alpha_n)^2}{B_n}\right)\|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{B_n} - \frac{2\alpha_n}{B_n}\Phi(\|x_{n+1} - p\|) \\
&\quad + \frac{\alpha_n^2 L(1+L)}{B_n}\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2.
\end{aligned}
\tag{2.6}
$$

From (2.6), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(1 + \frac{A_n}{B_n}\right)\|x_n - p\|^2 + 2\frac{\alpha_n \tau_n}{B_n} - 2\frac{\alpha_n}{B_n}\Phi(\|x_{n+1} - p\|) \\
&\quad + \frac{2\alpha_n^2 L(1+L)}{B_n}\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2.
\end{aligned}
\tag{2.7}
$$

But $B_n = 1 - 2\alpha_n k_n - \alpha_n^2 L(1+L) \to 1$, there exists a number $n_0 \in \mathbb{N}$ such that $\frac{1}{2} < B_n \leq 1$ for each $n \geq n_0$. From (2.7), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n \tau_n - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\
&\quad + 4\alpha_n^2 L(1+L)\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2.
\end{aligned}
\tag{2.8}
$$

$$
\|x_{n+1} - p\|^2 \leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n \tau_n + 4\alpha_n^2 L(1+L)\left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2.
\tag{2.9}
$$

The conditions $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$ and observe that $\sum_{n=1}^{\infty} A_n < \infty$. Hence, from (2.9) and Lemma 2.1, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. So that $\{x_n\}$ is bounded. Next, we set

$M_1 := \sup\{\|x_n - p\| : n \in \mathbb{N}\}$, $M_2 := \sup\{\alpha_n \tau_n : n \in \mathbb{N}\}$ and $M_3 := \sup\{\frac{a_n}{\alpha_n} : n \in \mathbb{N}\}$. Using (2.8), we have

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + 4M_2 - 2\alpha_n \Phi(\|x_{n+1} - p\|) + 4\alpha_n^2 L(1 + L)(M_1 + M_3)^2 + 2A_n M_1^2 \tag{2.10}$$

We now take $\theta_n = \|x_n - p\|$, $\lambda_n = 2\alpha_n$ and $\sigma_n = 4\alpha_n^2 L(1 + L)(M_1 + M_3)^2 + 2A_n M_1^2$, (2.10) reduces to

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n. \tag{2.11}$$

Using Lemma 1.9, it follows that $\|x_n - p\| \to 0$. The proof of Theorem 2.2 is completed. □

For the class of generalized strongly successively Φ-pseudocontractive mappings, the following results is immediate.

Corollary 2.3. Let C be a nonempty convex subset of a real Banach space E and $T : C \to C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and *generalized strongly successively Φ-pseudocontractive mapping* as defined in (1.8) and $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \in \mathbb{N}, \tag{2.12}$$

where $\{\alpha_n\}$ is a sequence in [0,1]. Assume that the following conditions are satisfied:
  (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$.
Then the sequence $\{x_n\}$ in C defined by (2.12) converges strongly to a fixed point of T.

Remark 2.4. Theorem 2.2 improves and generalizes the convergence results obtained by Huang [11] and several others in literature.

Example 2.5. Let $E = \mathbb{R}$ and $C = [0, 1]$. For all $x \in C$, we define $T : C \to C$ by

$$Tx = \begin{cases} (3 - \sqrt{x})^2 & \text{if } x \in [0, 1) \\ \\ 0 & \text{if } x = 1 \end{cases}$$

It is easy to see that T is *generalized strongly successively Φ-pseudocontractive mapping in the intermediate sense* with sequences $\{\tau_n\} = \frac{1}{n^2}$ and $\alpha_n = \frac{1}{n}$. Put $\Phi(t) = \frac{t^2}{3}$ for each $t \in [0, \infty)$
  We can see that the conditions (i) and (ii) of Theorem 2.2 are satisfied.

Competing Interests

The authors declare that they have no competing interests.

Authors' contributions

The first author conceived the study and computed most of the results while the second author computed others. All authors read and approved the final manuscript.

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