MATHEMATICS, AN EXACT SCIENCE: THE AXIOM OF CHOICE - A CASE IN POINT

BY

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IN DEED AND IN TRUTH

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INTRODUCTION

The Vice Chancellor,
Deputy Vice Chancellor (Management Services),
Deputy Vice Chancellor (Academic and Research Services),
The Registrar,
Dean of the Faculty of Science,
Other Deans,
Distinguished Guests,
My Colleagues,
Dear Students,
Members of the Press,
Distinguished Ladies and Gentlemen.

The Bible says in Romans 9[16] 'So then it is not of him that willeth, nor of him that runneth, but of God that showeth mercy." David says in Psalm 62 [11] "God hath spoken once, twice have I heard this, that power belongeth unto God."

I thank my Father, the Almighty God who has sustained me up to this moment.

Before I go further, I would like to state that this inaugural lecture is essentially in four parts. The first part attempts to answer the question "What is Mathematics?" The second part discusses sets, a popular term that many of us here are most likely to have heard about. The third part features the highlights of my research contributions and the fourth part appraises the state of mathematics education in Nigeria. In appraising the state of mathematics education in Nigeria, I have identified some problems militating against mathematics education in Nigeria and proffered some recommendations on the way forward.

'WHAT IS MATHEMATICS'

A Mathematician is often in a dilemma when asked to give a talk on Mathematics to the general public. Some years ago, Professor Sunday O. Iyahen, who is now the coordinator, Mathematics Programme, National Mathematical Centre, Abuja, gave a talk at the Nigerian Institute for International Affairs (NIIA), Victoria Island, Lagos. He made sure that in all his illustrations he did not go beyond Level I undergraduate Mathematics. Yet at the end of the lecture the first question/query from the audience was something like "How do you expect us to understand all the mathematics you have introduced?"

A similar situation can occur even when a mathematician gives a talk in his area of specialisation to his fellow practitioners. You can therefore appreciate the dilemma in which a pure mathematician whose area of specialisation is abstract mathematics may find himself when he has to give an inaugural lecture to a mixed audience. First non-mathematicians may misunderstand the technical terms used by a mathematician. For an English-speaking mathematician, for instance, the technical terms are often English words but not very plain English. They are such familiar and harmless words as group, continuous, compact, hereditary, etc., which you may feel you understand until you are told for instance that by a group we mean a set together with a binary operation satisfying certain stated conditions or that even though an undergraduate student of Mathematics may come into contact with the term "continuous function" in level II and be required to use it in subsequent years he may still not appreciate the ε–δ definition of a continuous function by the time he is in level IV.

A second reason is that in many science subjects like biology, agriculture, medicine, chemistry, engineering, many areas of physics and even mathematics, in an inaugural lecture of this nature, the audience may not understand the technical terms used but they will at least appreciate the fact that the lecturer's results have practical applications, may be for

the engineer in constructing more durable bridges, for the physician in curing certain ailments or for the farmer in producing more food etc. However, a pure mathematician, especially one doing research in abstract mathematics, may not be able to point to such practical applications of his work. Very often when a pure mathematician talks about the application of his work, he is not talking about how his work will lead to the construction of cheaper or more durable bridges by the engineer or how it can lead to production of more food or for any such mundane purpose; rather he is talking about how his result can be used to prove some other equally abstract theorems. However, this is not to say that we cannot find some practical use of the same result in another fifty to a hundred years. The problem at times is the gap between the pure mathematician and whoever is going to use his results. In any case, in appreciation of the role of pure mathematics in economic and technological development, the U.S., Russia and many other developed countries of the world have invested heavily on Pure Mathematics.

Over 90% of all mathematics we know today has been discovered in the last 120 years. The American Mathematical Society has classified mathematics into more than 60 subject areas. Out of these the average mathematician specialises in one area and may have interest in other five areas. This means the average mathematician may not have knowledge beyond an M.Sc. Level in more than six areas. What is more, he cannot claim to cover all sections of his area of specialisation.

Another thing is the depth to which a mathematician has to study his general area of specialisation. I took courses in Algebra, General Topology, Algebraic Topology, Functional Analysis and Complex Analysis for two sessions in the U.S in preparation for my Ph.D. Qualifying Examination, yet on no occasion, during that period, did any of my Professors find it profitable to refer us to journal articles. We were merely preparing for what some would call mathematics of the mathematicians – merely increasing our stock of vocabulary in terms of definitions, gathering information in terms of statements of theorems, and learning techniques in terms of proofs of theorems. Thus, the path of a research worker at the frontier of mathematical knowledge can be lonely, even his M.Sc. student cannot help him. Sylvester, one of the few eloquent Mathematicians once declared: "An eloquent Mathematician is as rare as a talking fish."

Many attempts have been made by eminent mathematicians to define mathematics. But first, I would like to give two examples to show what non-mathematicians may understand by the term mathematics.

When I was a secondary school student in the 1950s, we heard stories about how one Dr. Chike Obi - he was the first Nigerian to obtain Ph.D. Mathematics – would visit a secondary school, enter a classroom, ask a student to stand up and then say, for instance, "You are 5ft. 1¼ inches tall and your weight is 84lb 7¾ oz.

The second example was that sometime ago I had a discussion with some young people. One of them wanted to know the total amount of subsidy per day the Federal Government claimed it was paying on 17m litres a day at the rate of N12.00 per litre. As one of them was about to use his calculator, I gave the answer as N204m. Instead of saluting me, "E ku aigbagbe" (Yorubas' greeting for exhibiting a retentive memory) their reaction was like after all he is a professor of mathematics. Now I was sure I would have performed the same operation as fast when I was in primary school. Thus, while to some a mathematician is one who has almost magical powers to determine exact lengths and weights yet to others he is supposed to be a calculating machine. Here are some attempts made by mathematicians to define mathematics. [1].

- B.W. Russell (1901) "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true".

- B. Peirce (1881) "Mathematics is the science which draws necessary conclusions".
- F. Klein (1829-1929) "Mathematics, in general, is fundamentally the science of self-evident things".
- D. Hilbert (1862-1943) "Mathematics is nothing more than a game played according to certain simple rules with meaningless marks on paper"
- B.W. Russell (1903) "Pure mathematics is the class of all propositions of the form "p implies q", p and q being carefully specified".

Applied mathematicians, on the other hand, could say that mathematics is the language of experimental physics.

You can see from these and many other definitions that trying to define mathematics is like a hopeless attempt at trying to paint a brilliant sunrise in one colour. However, it is generally agreed that the real objects of mathematics are arrays of forms, deployed in a network of formal rules, formal functions, formal axiom systems, explicit theorems together with their careful proofs and the manifold interconnections of these forms [5]. Thus in a mathematical treatise we have essentially a sequence of theorems, each followed by its proofs. The proof of each theorem is logically inferred from previous theorems. But this means that the first "theorem" we put down cannot be proved. The unprovable theorems which can be regarded as assumptions and which initiate a theory are called axioms (or postulates). The truth of axioms cannot be established by inference since inference presupposes them, or by observation, which can never establish necessary truths. So they are held to be objects of intuition. Axioms are ordinarily truisms, consequently self-evidence is taken as a mark of intuition. I would like to point out that it does not matter whether the axioms are true or false. The theory merely says that the subsequent theorems are true provided that the axioms that initiate the theorems are true.

Moreover, to pure Mathematicians it is irrelevant what the objects are that enter into the axioms, such as points, lines or circles. What matter are the relations between such objects, for instance, that two straight lines intersect at not more than one point. This makes the theory most useful. For whenever objects satisfying the relations asserted in the axioms are encountered, for instance, in physics, the entire theory can be taken over and applied to the objects [3].

Notice that the fact that it is irrelevant whether the axioms that initiate a theory are true or false does not mean that anybody can just state a set of axioms and begin a theory. To be acceptable, a set of axioms should possess the following three properties:-

1. Completeness: This means that everything that will be used in the theory is set down in the axioms so that there are no tacit assumptions;

2. Consistency: This means that it is impossible to derive two contradictory theorems from the axioms. In particular, the set of axioms must not be such that its acceptance will invalidate other previously established results; and

3. Independence: This means that none of the axioms car be logically inferred from the others.

An axiomatic approach is the most efficient way to introduce algebra. Once a body of theorems has been deduced from axioms, we know that the theorems hold for every structure that satisfies the axioms. For example, if we prove a theorem just in terms of the axioms of a ring without using any other properties, then this single proof allows us to use the theorem freely for any ring.

Mathematics as a Science, Philosophy and Art

In the last 120 years, mathematics has taken the form of a Science, a philosophy and an art.[3].

1. As a Science, Mathematics has been adapted to the description of natural phenomena. Here, mathematics is regarded as an intellectual machine which works successfully.

2. As a philosophy (in the sense of Whitehead, 1929), there is the endeavour to frame a coherent, logical and necessary system of general ideas in terms of which every element of mathematics can be interpreted. Examples are the careful analyses of mathematical concepts, the formulation of rigorous canons of inference and proof, the search for consistency and the development of logical analysis.

3. As an art, Mathematics is a whole world of invention and discovery. Just like that of a poet or a painter, the creative imagination of a mathematician leads to the construction of a new theorem, the intuition of some new principle, or the initiation of a new branch of mathematics.

Sets

We are now in a position to state what informs the choice of the title of this Inaugural Lecture. Whether consciously or unconsciously we all apply the knowledge of sets. We talk of a set of people, a set of houses, a set of schools, etc. For instance, Nigeria is a member of the set of countries in Africa. Perhaps it is in the realisation of the necessity to have a knowledge of sets that the study of sets has been introduced into our primary schools. I shall come to that later. Mathematicians are aware that set theory is an important foundation for contemporary mathematics and sets form a powerful language for reasoning about mathematical objects. Furthermore, mathematicians have carefully enunciated an axiomatic theory of sets.

Now, Mathematics has been popularly referred to as an exact science. That is, a science that uses accurate measurements and set rules. In view of this definition one may be tempted to ask whether mathematicians ever disagree on issues that concern, say, a theorem or its proof. It is an attempt to answer this question that leads to the following discussion of sets and subsequently to the Axiom of Choice taken to be one of the axioms of set theory and which in turn informs the choice of the title of this inaugural lecture, which is: "Mathematics, an Exact Science: The Axiom of Choice – A Case in Point."

Two sets S and T are said to be equivalent (or to be of the same power) and we write S ~ T if the elements of the two sets are in one-to-one correspondence. Every set which is empty or has *n* elements, where *n* is a natural number, is called a finite set. Any other set is called an infinite set. We shall refer to this as the usual definition of a finite set.

There are examples of sets which we cannot classify as finite or infinite. For instance, the set of all prime numbers of the form 2^n+1, where *n* is a natural number, and the set of all prime numbers *p* for which p+2 is also a prime number. Since the union of two finite sets is a finite set it follows that if the set S is infinite and S=AUB, then at least one of the sets A and B is infinite. However, even in the decomposition of the set N of natural numbers we may not be able to determine which part is infinite. A case in point is when A denotes the set of all natural numbers *n* for which the number $2^{2^n}+1$ is a prime and B = N\A. Here, all we know is that at least one of the sets A and B must be infinite.

In the same way we may not be able to determine whether two sets are equivalent or not. For example, we are not able to determine whether the set of all prime numbers of the form 2^n+1, where *n* is a natural number, is equivalent to the set of all natural numbers. In order to prove that two given sets S and T are equivalent, it is necessary and sufficient to prove that there exists a one-to-one correspondence between the elements of the set S and those of the set T. However, to prove the

existence of such a correspondence, it is not necessary to establish it or to give an example of it. Its existence may be deduced from the axioms assumed and theorems proved previously and we may even give an indirect proof.

Existence Theorems

Before going further, let us briefly consider Existence Theorems [6].

An existence theorem shows only that some mathematical entity satisfies a particular condition. The proof may be indirect and need not actually produce the entity but it must show that there is such a solution. Some Mathematicians philosophically object to existence theorems or existence proofs because of their non-constructivity. They demand a "constructive proof" instead. A constructive proof not only shows that one (or more) mathematical entity satisfies a particular condition, but also actually produces it (or them). As a compromise, these Mathematicians, often referred to as intuitionists, demand a certain limited kind of construction proof but they reject proof by contradiction.

A proof by contradiction, also known as *'reductio ad absurdum'* (reduction to absurdity) proves a theorem by assuming that the conjecture is false and finding that it leads to a contradiction. When a contradiction is found the conjecture is then proved to be true. *Reductio ad absurdum* is an "indirect proof", the opposite of "direct proof". A direct proof satisfies the most stringent requirements. It follows a step-by-step process based on axioms or other theorems, especially theorems that are also proved by direct means. However, it is agreed that a direct proof is often not practical, too often not profitable and sometimes not possible. Thus, if we were to follow the intuitionists much of mathematics would not get done.

THE AXIOM OF CHOICE

In 1904, E. Zermelo stated the following:

AXIOM OF CHOICE. Given any non empty family $\{A_\alpha | \alpha \in Æ\}$ of non-empty pairwise disjoint sets, there exists a set S consisting of exactly one element from each A_α.

It has been proved that the axiom of choice is equivalent to the following two statements:

1. Zorn's Lemma. Let X be a pre-ordered set. If each chain in X has an upper bound, then X has at least one maximal element.

2. Zermelo's Theorem. Every set can be well-ordered. [2].

Before I go into the controversy on the axiom of choice let me deal briefly with the controversy concerning the true author of Zorn's Lemma. It happens at times that two mathematicians may obtain the same result at about the same time independently of each other. Thus, even though it was Zorn's result that was first published, Polish mathematicians insist that it was K. Kuratowski that first obtained the result. As a result of this if you read a Polish text, what is stated above as Zorn's Lemma would be stated as Kuratowski – Zorn's Lemma. The story goes that the first time these two authors met at a conference, a mutual friend who introduced them to each other started by saying Professor Kuratowski, this is Professor Zorn of the Zorn's Lemma fame. Then he turned to Professor Zorn and said Professor Zorn, this is Professor Kuratowski, the author of Zorn's Lemma.

Now let us go back to the Axiom of Choice. Zorn's Lemma is a particularly useful version of the Axiom of Choice. It is applicable to existence theorems whenever the underlying set is partially ordered and the required object is characterised by maximality. When we apply the Axiom of Choice to a denumerable infinity of sets, we say that we apply the restricted Axiom of Choice.

Notice that this axiom does not contradict intuition or other accepted axioms, so we can either accept it or reject it. Here, we notice that the rejection of an axiom does not mean the acceptance of the negation of the axiom[8]. We also make the following observations as regards the Axiom of Choice

1. A large number of particular cases of this axiom has been proved independently of it.

2. None of the many conclusions which have been drawn from the Axiom of Choice has lead to a contradiction.

3. The Axiom of Choice simplifies, considerably, various parts of the theory of sets and of the calculus and is indispensable for proving many important theorems of those theories.

The Axiom of Choice is the only existential axiom. Unlike the other axioms of the theory of sets, a set obtained by application of this axiom is not in general, uniquely determined by the given conditions.

In 1938, K. Godel proved that if the set theory based on the other generally accepted axioms of the theory of sets is consistent, then the set theory based on the axiom of choice together with these other axioms is consistent. Godel's result however leaves open the possibility that the axiom of choice is derivable from the other axioms. In 1963, P. J. Cohen proved that it is not. It follows that the axiom of choice is an independent axiom. Before Godel gave his proof, N. Lusin's comment on the axiom of choice was: "For me the proof of a theorem by means of the axiom of choice is an indication that the individual giving such a proof agrees that it is a waste of time to give an exact proof or the falsity of the theorem in question."

While many mathematicians, including Zermelo himself, assert that this axiom is as obvious as the other axioms and that it can be accepted without hesitation, others say there is no reason to believe that the axiom is true. G. Peano who also proved the independence of the axiom is neutral as he maintains that the question of obviousness is not a logical but a philosophical problem.

B. Russell is of the opinion that the axiom of choice ceases to be obvious once its meaning is understood. He further states that it may be true but that it lacks obviousness and that the conclusions drawn from it are astonishing. He feels it would be advisable to avoid using it.

D. Hilbert states that the Axiom of Choice is based on a general logical principle, necessary and indispensable for the very foundation of mathematical deductions.

In opposing those who say that the Axiom of Choice should be rejected, A Fraenkel points out that the Axiom of Choice has been introduced in the same way as the other axioms of mathematics, which have been obtained by the analyses of known processes of reasoning. He claimed that, on these lines, Greek mathematics was lead to the axiom of parallels which was accepted although there was not, and as we know today, there cannot be any proof of it. And even now, when we know that the axiom of the parallels could be rejected, nobody thinks of rejecting it or of abandoning the further development of Euclidean geometry, based on this axiom. He further argues that, similarly, it would not be justifiable to reject those branches of mathematics which are based on the Axiom of Choice, as to do so would radically restrict the Set Theory by deleting several very important parts of it.
Consider the notion of Lebesque measurable subsets of the real line. Like every other such notion in mathematics, if either every subset of the real line or no subset of the real line is Lebesque measurable, then the notion is not worth studying. It turns out that all subsets of the real

line which can be easily visualised are Lebesque measurable and even though the set of non measurable subsets of the real line is uncountable, I do not know of any that has been given without the aid of the Axiom of Choice.

J. König points out the obvious that the Axiom of Choice is not an axiom in the usual sense of the word. This arises from the fact that we can argue whether it is true or false. Many mathematicians would agree that the following two statements are logically equivalent:

(1) There exists an object having a given property and

(2) It is not true that no object has the property in question.

The so-called intuitionists, however, do not share this view. According to them, if all we know about a given set is that it is not empty, then we cannot pretend to be able to define one of its elements in such a way that two persons would be thinking of the same element or to be able to choose one element of that set.

W. Sierpinski feels that the main reason why some mathematicians reject the axiom of Zermelo is that it is called the Axiom of Choice. He argues that by accepting Zermelo's axiom we say nothing about the possibility of choosing one element from each subset belonging to the given family of subsets

Zermelo himself acknowledges that this proposition as well as the name "Axiom of Choice," concerns only the psychological method of presentation, while the axiom should be regarded as a pure axiom of existence. Unfortunately, as we had pointed out earlier on, the identification of mathematical existence with constructibility is the fundamental principle of intuitionism. The reason why the majority of intuitionists of a radical and conservative type reject the Axiom of Choice is because of the existential and non-constructive nature.

If we ask ourselves whether the Axiom of Choice is true or not we would encounter further difficulty, namely, the definition of mathematical truth. In 1900, D. Hilbert, at the International Congress of Mathematicians in Paris, challenged his colleagues to discover a precise method for determining the truth (or falseness) of any given statement in formal logic. However, B. Russel in 1901 and K. Godel later showed that such a method for determining the truth could not be found.

The simplest case of the Axiom of Choice is that in which the family of sets consists of a single set A. Then the Axiom of Choice is reduced to the statement that if the set A is non-empty, then there exists at least one element of the set A. This statement is true since the statements, "the set A is non-empty" and "there exists at least one element of the set A" are equivalent. Now we are not asserting that we can indicate one element in every non-empty set, or that we can choose a particular element from every such set, rather, we are merely asserting the existence of such an element.

Another example is when each set of the family consists of two elements. Thus, let $F = \{A_\alpha \mid \alpha \varepsilon Æ\}$, where $A_\alpha = \{P_\alpha, Q_\alpha\}$ form a partition of the set $\Re$ of all real numbers. *Then, the required choice function f is given by $f(A_\alpha)$ is that one of the sets* P_α and Q_α which contains the number 0 as an element. However, if $F = \{A_\alpha \mid \alpha \varepsilon Æ\}$, where $A_\alpha = \{P_\alpha, Q_\alpha\}$, P_α and Q_α are denumerable subsets of $\Re$ and $P_\alpha \cap Q_\alpha = Ø$, then, we are not able to define a function *f* which associates with every set $A_\alpha \varepsilon F$ a particular element of A_α.

We now go back to the definition of a finite set.

Some mathematicians would like to develop the theory of finite sets without the aid of concepts or theorems of arithmetic of natural numbers. In this direction, A. Tarski in his paper published in 1924 defined a finite set as follows:

"The set S is finite if in every non-empty family F of its subsets there exists a subset $T \varepsilon F$ such that no proper subset of T is a member of F." Another definition of finite set due to Dedekind is: "Every set which is not equivalent to a proper subset of itself is finite and every other set is infinite." These two definitions due to Tarski and Dedekind, respectively, have the advantage of not being based on the concept of natural numbers. The problem is that to prove that Tarski's definition is equivalent to the usual definition we have to apply the well-ordering principle while we cannot show that Dedekind's definition is equivalent to the usual definition without the aid of the Axiom of Choice.

H. Poincare [8] ridicules those who would like to define the concept of a finite set in these two ways as follows: "Many mathematicians have followed this way and have asked many questions of the same kind. They are so familiar with transfinite numbers that they succeed in making the theory of finite numbers dependent on that of the cardinal numbers of Cantor. In their view, to teach arithmetic in a really logical manner, one should begin by establishing general properties of transfinite cardinal numbers and then distinguish among them, a small class, the class of ordinary integers. In this way, one could arrive at proving propositions relative to this small class (i.e., any arithmetic property and algebraic property) without using a principle foreign to logic.

SETS OF THE POWER OF THE CONTINUUM

A set which is equivalent to the set of all real numbers is said to be of the power of the continuum. Now, hear the intuitionists. According to Brouwer, the set of all real numbers is not a finished creation. Its elements cannot be said to exist but only to be coming into existence in the process of our defining them. Thus, the set of all real numbers is presented as a set in which we are able to indicate every time, only a denumerable subset, but for every subset of this type it is always possible to define the real numbers which do not belong to it.

N. Lusin, on his part, questions the concept of the set of all real numbers since we do not have a general law defining every real number. Brouwer goes further to reject the concept of all natural numbers.

A. Fraenkel in his 1927 paper however is of the opinion that a theory of sets which did not ensure the existence of sets of the power of the continuum so essential for both Analysis and Geometry would be a mere shadow and would mean a suicidal renunciation of the techniques of Cantor for the sake of intuitionism.

MY RESEARCH CONTRIBUTIONS

My field of specialisation is Semigroup Theory and my research has been in the areas of binary relations on topological spaces and on dendrites. Even though some scattered remarks regarding the concept of relations are to be found in the writing of medieval logicians, C.S. Peirce through several papers published between 1870 and 1882, is regarded as the creator of the theory of relations.

I am however concerned with a special case of this general theory, which is the theory of the semigroup of binary relations on a topological space. The theory of the semigroup of binary relations on a set is the abstract study as well as a generalisation of partial transformations on a set. Work on endomorphisms of binary relations on a set seems to have dated back to the work of I. Schreier in 1937. However, virtually all the work done on binary relations on a set before the last forty-five years have been restricted to special types of binary relations, such as one-to-one partial transformations. Much work has been done on semigroups of partial transformations and binary relations by V.V. Vagner, D. Rees, E.S. Lyapin, K.A Zarecki and others. K.D. Magill and S. Yamamuro were the first to consider binary relations on a topological space. Their work was based on the work of A.H. Clifford and D.D. Miller on binary relations on a set.

My publications have been on Hom_{CM} (C[X], C[Y]), the non-constant homomorphisms from C[(X] into C[Y], where C[S] is the semigroup of all closed relations on a C-space S. The highlights are 1976 (*Semigroup Forum* 13. pp 1-17), where I completely characterised the maximal and minimal elements of Hom_{CM} (C[X], C[Y]); 1977 (*Semigroup Forum* Vol. 14. pp. 331-353) where I determined when two elements of Hom_{CM} (C[X], C[Y]) have a joint as well as when they have a meet. 1978 (Semigroup Forum Vol.16 pp 403-425), where I generalised Magill's result on M[I], where I is the closed unit interval, to M[Y], where Y is any hereditarily locally connected continuum. I also worked on the structure of β_x the semigroup of binary relations on a set X. 1986 (*Demonstratio Mathematica*. Poland, Vol. XIX No 4 pp 895-913), where I characterised Green's relations, including the J-relation for the first time, on β_x by direct composition of the relations, by introducing the notion of a skeleton. Before then, R.J. Plemmons and M.T. West had characterised Green's relations on β_x but without the J- relation by interpreting a relation as a Boolean matrix. I have also worked on dendrites 1981 *(Geometriae Dedicata Erlagen*, 27 pp. 227-240), where I calculated the number, up to homoemorphism, of certain types of dendrites of order three and 1989 *(Geometriae Dedicata*, 32 pp. 255-264), where I characterised accessible simple B-finite dendrites of finite order and also accessible saws and 1997-98 *(Journal Sci. Res. Dev.* Vol 3 pp 95-102), where P.H. Fisher and I calculated the number of Semi simple saws. I am now trying to characterise homomorphisms from C[X] into C[Y] that are not necessarily union and symmetry preserving.

STATE OF MATHEMATICS EDUCATION IN NIGERIA

The use of mathematics in any society cannot be over-emphasised. The knowledge of mathematics has great effect on the daily activities of man and in the organisation of the rational society in which there will be leisure for the vast majority. The knowledge of mathematics is needed in the Sciences, Engineering, Technology, Medicine, Agriculture, the Social Sciences, Philosophy, Architecture, Linguistics, Law and Religion.

It is in realisation of this that General Mathematics is made compulsory at SSCE level. In the same vein, a credit pass in mathematics is required for admission into virtually every department in the University of Lagos.

At the basic level, mathematics consists of arithmetic, algebra, geometry and trigonometry. I am not unaware of the fact that sets is being taught in our primary schools now. Fortunately, virtually everybody can understand the basic mathematics if given the right atmosphere.

In April 2000, a document entitled "Principles and Standards for School Mathematics" (PSSM) was published in the U.S. It is an updated version of the school mathematics standards issued by the National Council of Teachers of Mathematics (NCTM) over several years starting from 1989. The Notices of the American Mathematical Society (NAMS) invited four individuals to write short pieces, describing their reactions to Principles and Standards. I take some excerpts from the reactions of each of two of them to illustrate my views [7]:

1. Susan Addington: "... But reforming K-12 Mathematics education in the US is like turning the Titanic around with a canoe paddle or perhaps like herding a million cats . . . Much of the vitriol in the "math wars" that broke out in the 1990s comes from deeply held beliefs about mathematics education: that traditional Euclidean geometry is the best, or the worst, way to teach mathematical reasoning; that paper-and-pencil calculation is the key to, or a major impediment to understanding numbers; that calculators should not be used until paper-and-pencil arithmetic has been mastered or that they should replace paper-and-pencil arithmetic. Mathematics education research do not give definitive answers to these questions, and many thoughtful teachers would agree with parts of each opinion."

I am of the opinion that, these days, it will be unrealistic to avoid the use of calculators. However, I am always disturbed when a

mathematics undergraduate student has to reach out for his calculator to obtain answer to say, 7 x 9 or 4 x 8. It is not unusual for an undergraduate to write down ¾ = 0.43, or something like that. Of course, his explanation would be that he used his calculator. A case of garbage in garbage out you would say.

2. Herbert Clemens: "Let us be very careful about asking teachers for more than we have any right to expect. We must not foist upon them the responsibility for remedying a deficiency that we as a society cause by the reward structure we have designed and perpetuated. It would be better for all concerned if high school did a lot less in the way of advanced placement calculus courses and spend more time on deepening understanding of "elementary" mathematics like geometry, algebra and arithmetic. This change would be especially beneficial for "mathematically gifted" students."

At the conference of the Mathematical Association of Nigeria (MAN), which took place in Port Harcourt in 1976, a number of us objected to the idea of introducing the so-called Modern Mathematics into our Secondary Schools. Some professional authors who had already published modern mathematics textbooks attended the conference to advertise their wares. After some heated debate on the matter it was agreed that Modern Mathematics as a subject in West African School Certificate Examination (WASCE) should be dropped; and West African Examination Council (WAEC) agreed to drop it.

I was introduced to Modern Mathematics as a subject in 1962 at the University of Ibadan. I had expected that the syllabus would be a continuation of what I had done up to the General Certificate of Education (GCE) Advanced Level –harder problems and more elaborate calculations. That was not to be. In fact, within four weeks I gave out my four-figure table to a pre-medical student since I felt I would not need it anymore. However, despite the suddenness with which we were introduced to modern mathematics, a good number of us were able to cope pretty well.

I would like to make an observation here. Before WAEC dropped Modern Mathematics as a subject in WASCE, going through the syllabus as well as the question papers, I was sure that within my first four to six months at the University of Ibadan I had covered the syllabus at a greater depth than required at WASC level, whereas only very few of the students who had started Modern Mathematics from Form I had an idea of what it was all about by the time they were in Form V. This was not surprising since neither the students nor their teachers were prepared for the introduction of Modern Mathematics. I came in contact with Economics as a subject for the first time in February 1960. Yet, despite the fact that I was working full time, I took Economics together with Pure Mathematics and Applied Mathematics at the GCE Advanced Level Examination eleven months later, in January 1961 and passed all the three subjects.

The point we are making here is that it is more profitable to concentrate on a few core subjects and study them well than to take many subjects and end up not studying any of them well. The developed world seems be developing so fast for us that we tend to lose our focus. From the primary to tertiary level of education, we seem to feel that the way to solving our problems is to introduce more courses without thinking of overloading the students or the poorly paid, ill motivated teachers. In my own view, it is counter-productive and unfair to make a thirteen-year-old to study as many as 15 subjects and take some 12 of them at the JSS 3 Examination.

Problems Militating Against the Study of Mathematics

The problems militating against the study of mathematics at the various levels of our educational system are well known, especially to mathematics teachers and they have been discussed at different fora. The following are a few of such problems:

1. The students, especially at the secondary school level, have a negative attitude towards the subject. They regard mathematics as a difficult subject and do not exert themselves to try to understand the subject;

2. There are too few qualified and experienced mathematics teachers. For any teacher, a good knowledge of the subject he teaches as well as the knowledge of the curriculum is necessary. A Latin dictum says, *"Nemo dat quid non habet"* (Nobody gives what he does not have). Unfortunately, because of the falling standard of education in the country, the competence of many of those teachers who even have the paper qualifications is in doubt.

   When we talk about falling standard of education we do not mean that none of our graduates meet up the standards. There are definitely those who can hold their own anywhere in the world. My people say, *"Ba gun 'yan ninu ewe ba se 'be 'nu epo epa eni mayo a yo"*. (However inadequate the food supplies for a society may be, some members will still have enough to fill their stomach). For instance, there was a case in the 1940s where the only candidate who passed the Senior Cambridge Examination in his school that year obtained Grade I with six Distinctions. Again we are not unaware that there are a few government and private secondary schools including the International School, University of Lagos, where standards are high.

Nevertheless, the proportion of NCE or university mathematics graduates that is knowledgeable enough to effectively teach mathematics at the secondary school level is on the decrease. Those who argue that the standard of education has not fallen would like to point out, for instance, that primary school pupils can now tell whether it is the tea cup that normally stands on the saucer or the other way round, and what is more, that kids of nowadays can even play games on the computer, two feats that I could not perform at the secondary school level.

In the *Punch* newspaper of 3rd October 2003 it was reported:

"A study conducted by the West African Examinations Council has found that its examiners lacked rater and inter-rater ability in the marking of essay papers of Senior Secondary Certificate Examination, a situation that could be traced to the subjective nature of essay tests.

Rater entails the ability of examiners to interpret and adhere strictly to examination marking schemes..."

From my own experience from the 1980s up to 1992, a number of WAEC Assistant Examiners are not knowledgeable enough to perform effectively and efficiently as examiners at that level. Yet, some of those inefficient Assistant Examiners of the 1980s have now become Team Leaders who are now supposed to vet the scripts marked by the new generation of Assistant Examiners. In our primary schools, there are cases where the notes of lesson of some of the mass produced NCE teachers are so full of grammatical errors that their headmasters did not bother to correct such errors but simply read through to have an idea and then sign. In some cases, such headmasters happen to be old Grade II teachers. One wonders if such NCE teachers can be said to understand method of teaching

3. The society is not adequately informed about mathematics. Many regard mathematics, even at the secondary school level, as one of those esoteric subjects. In fact, one of the reasons hindering technological development in Nigeria is the gap between practitioners and the general public.

4. Most parents are not in a position to help their children.

5. Another problem is the imposed medium of communication. Where the language is foreign children may mimic adults without understanding. The renowned educationist, Professor A. B. Fafunwa (NNOM) in a keynote address which he delivered at the Ajasin Foundation 3rd Annual Colloquium in November 2002 at the Chartered Institute of Bankers, Victoria Island, Lagos stated as follows:

"One of the important factors militating against the dissemination of knowledge and skills and therefore of rapid social and economic well-being of the majority of the people in developing countries is the imposed medium of communication. There seems to be a correlation between under-development and the use of foreign language as the official language of a given country in Africa, e.g., English, French or Portuguese. It is my belief that a nation that uses another nation's language and abandons its own cannot grow fast. It certainly slows down its national development until it elevates its own languages for economic, scientific and political activities."

In the case of mathematics the problem becomes compounded. On the one hand, there is the use of mathematics language and on the other hand there is the use of the medium of communication which is different from the mother tongue. Perhaps it is in realisation of this that UNESCO has for some time been pioneering and promoting the use of mother tongues as a medium of education. In Europe, for instance, even countries whose populations are less than five million use their own respective languages.

6. It is obvious that funding and corruption are the greatest obstacles militating against the development of mathematics education in

Nigeria. Our education at all levels is critically under funded. Even though governments at the various levels would like to be seen to be trying their best at funding education, I believe that they can still do much more. However, it must be obvious by now that to fund our universities adequately we cannot afford to depend on government sources alone.

At the end of the apartheid era in South Africa, some perceived the black South Africans as being in the fourth world. Yet when the then President Nelson Mandela, was asked when university education in South Africa would be free, his answer was brisk and direct "Not now, not in the nearest future."

I would like to quote part of an article, "The Cost of Excellence", which the Provost, Elizabeth D. Capaldi sent to the Winter 2003 Edition of UB-today, a publication of the University At Buffalo Alumni Association. She declared:

> "Excellence is costly but it is imperative that we achieve it at UB. We must maintain our upward trajectory toward the highest level of excellence, in all our classrooms, in every public service initiative we undertake. This means we must generate funds to pay for the level of achievement we aspire to and deserve. We are continuing to work in ways of producing our own funds, recognizing that New York can only do so much in the face of declining state revenue, and given other state priorities ... State universities are now commonly referred to as state-supported or state-assisted institutions, in recognition of the fact that the majority of campus income now comes from non-state sources."

Perhaps it would be instructive to point out that by 1976 the Government of the late Governor Rockie Feller spent $2 bn on a new Amherst Campus for UB. Most of us know that there is no free university education in the U.S. Education should be one of the priorities of our various governments. At the same time, provision of infrastructures should be a priority since those who complete their education at various levels should be gainfully employed. Thus, it should be obvious that, judging from the level of our economy, Nigeria cannot afford free university education. After all, students pay fees in both private and State universities. What we should be talking about is how to reduce the burden on brilliant but poor students.

We all know that corruption has eaten deep into the fabrics of the Nigerian society and that it is much pronounced at the top level of our various governments. This may account for why Nigerians are always sceptical about every move of our governments. Nevertheless I would like to point out that the average Nigerian expects government to provide amenities but he is unwilling to contribute his quota. The average Nigerian elite is like a Christian who wants the Pentecost but not the Calvary. What is more, our politicians are always ready to pander to the wishes of the people. We started with Free Primary Education which the Government of the late sage, Chief Obafemi Awolowo introduced and which was a well conceived and, initially, a well executed programme. Just when we started making a mess of it, other politicians started promising free and qualitative education. One would have thought that free education is the antithesis of qualitative education, but trust our politicians.

7. Inadequate staffing: As a result of poor funding, the universities have not been able to retain the best academic personnel. These days most students work hard to be able to get jobs in financial institutions and oil industries. Of course, many of them also want to be offered admission into universities in developed countries. Many of those in this group feel that once they "escape" from

Nigeria, they are not likely to come back to take up appointments after completing their postgraduate programmes unless they secure high paying jobs preferably one in which they are likely to have the opportunity to travel regularly to the developed countries.

Recommendations

Mr. Vice-Chancellor Sir, various groups and individuals have made recommendations regarding the improvement of the state of Mathematics Education in Nigeria. I particularly feel reluctant to make recommendations on this occasion because I feel that our problem is not that of lack of knowledge of what to do but of the means and will to do them. I was one of the rapporteurs at the Second Pan-African Congress of Mathematics under the auspices of the African Mathematical Union, held in Jos, Nigeria in March, 1986. On that occasion, for instance, following a critical review of the problems and challenges facing the study and teaching of Mathematics, as well as research in Mathematics in the African Continent, we made eighteen recommendations. In realisation of this, I would like to make the following few recommendations:

1. As we had mentioned earlier, it is obvious that inadequate funding and corruption are the greatest obstacles militating against the development of Mathematics education in Nigeria. We recognise the effort of the Federal Government through the Nigerian Educational Research Council (NERDC) at creating specialist teachers of Science and Mathematics in Primary Schools, the introduction of an annual Teacher Vacation Course (TVC) for Secondary School Mathematics and Science Teachers by the Federal Ministry of Education in conjunction with all the State Ministries of Education, and the establishment, by the Federal Government, of the National Mathematical Centre (NMC), Abuja, for the purpose of solving the problems besetting the teaching and learning of Mathematics, Mathematics Education, Statistics, Computer Science and Theoretical Physics at various levels of

our educational system [4]. However, for lack of adequate funding and corruption, all this effort has not yielded the desired fruits. In the primary schools, for instance, a teacher still has to teach all subjects to his class. So, even teachers who are afraid of or hate Mathematics, apparently because they do not have the required knowledge, are compelled to teach the subject.

We commend the effort of the Lagos State Government at moving in the desired direction in the sense that specialist Mathematics teachers now teach Mathematics in Primary IV through Primary VI in public schools. This arrangement does not necessarily require more funds. We complicate matters by insisting on training specialist teachers before carrying out this policy. The "specialist" teachers are already available. Each state or local government together with the headmasters of the various primary schools should simply select the teachers who can most effectively teach Mathematics in their respective schools. Of course, this may result in the transfer of some teachers from one school to the other.

With the foregoing analysis, I recommend that the Federal Government should take the initiative to invite other stakeholders - the State and Local Governments, the Universities, parents, employers of labour and various donor agencies - to fashion out ways of sourcing funds from non-governmental sources. We are happy to note that in terms of generating funds from non-governmental sources, the University of Lagos is blazing a trail through the effort of the Pro-Chancellor and Chairman of Council, Chief Afe Babalola, SAN, OFR, LLD and our indefatigable Vice-Chancellor, Professor Oye Ibidapo-Obe. The recently commissioned Engineering Laboratory, donated by Julius Berger PLC is an example.

2. Federal and State governments should, as a matter of policy, provide special incentives for science teachers at both secondary and tertiary levels. I know that secondary school teachers who were science graduates were being paid an incentive allowance of £10 (ten pounds) per month by the Western Nigerian Government sometime in the 1960s. This amount was the equivalent of about 16.7% of a fresh graduate teacher's monthly salary at that time.

3. All effort should be made to make both students and parents realise early enough that a student who fails to obtain a credit pass in Mathematics at the Senior Secondary Certificate Examination (SSCE) has very few options both at the university level and vocational training.

4. Federal and State governments and wealthy individuals should give scholarship and Bursary awards to Mathematics and Mathematics education students. We commend the effort of the Federal Government in this direction, whereby students who obtain a minimum CGPA of 3.00 are already enjoying Bursary awards and we recommend that this minimum be lowered to 2.50 for Mathematics and Mathematics Education students.

5. It cannot be over-emphasised that Mathematics is not only the basis for all scientific development but also the key to understanding various other subjects. Indeed from my own observations I can say with confidence: "Show me a young man who at one sitting makes a credit pass or above in each of General Mathematics and English Language at the SSCE and I will show you a young man who makes at least five credit passes or above in the same examination". With this realisation, I recommend that more hours be devoted to the teaching of Mathematics in both primary and secondary schools.

6. From my observation, if a fresh school leaver who obtains not less than A3 pass or its equivalent at the SSCE were to teach an SS3 class for just a year, the result of the class at SSCE would be better than if a mathematics education graduate, who obtained equivalent of C6 when he took the SSCE, were to teach the same class. The fact is that at the university level, even though students are required to use the knowledge of the general mathematics learnt at the secondary school level, the university mathematics syllabus does not make room for the revision of general mathematics. Thus, the young man who could not solve certain problems under, say quadratic equations, by the time he completed his secondary school education, is not likely to be able to solve such problems even after obtaining his degree in Mathematics unless he recognises and effectively works on such deficiencies. I, therefore, recommend that for mathematics education undergraduate, the Level I / Year I Mathematics syllabus be devoted to the revision of General Mathematics and Further Mathematics and that they do not go beyond Level III / Year III Mathematics, for the purpose of their Mathematics Education degree.

7. Many members of the academic community have observed that a situation whereby University admissions are based solely on the results of a four hour examination conducted by JAMB and the SSCE results are used only for the purpose of eligibility for registration is undesirable. In fact, I do not see any reason why the SSCE results should not be more reliable than the UME results. I, therefore, strongly recommend to the Federal Ministry of Education that both the SSCE and UME results be used for the purpose of admissions into our Universities and that each university that so desires be allowed to conduct interviews for its candidates.

8. MAN Competition. To encourage our young ones to develop a positive attitude towards the study of Mathematics, the Mathematical Association of Nigeria (MAN) in collaboration with the National Mathematical Centre (NMC), Abuja, organises an annual Mathematics Olympiad across the country. To do this, each State Branch of MAN first organises her own Mathematics Olympiad to select those to take part at the National level. From the year 2001 to date I have been the co-ordinator of the Mathematics Olympiad conducted for Primary 6, JSS 3 and SS 2 by the Lagos State Branch of MAN. The schools complain that there is no provision for the funding of this laudable project. In fact there are cases where some dedicated Mathematics teachers had to pay for transportation of their candidates to the examination venue. I am hereby appealing to both the Federal Ministry of Education and the State Ministries of Education, especially the Lagos State Ministry of Education, to make funds available to their schools for this laudable project.

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It is with a heavy heart that I remember the death of the late Alhaji Dauda Ademola Agbalaya, the late Secretary of the Housing Committee, which occurred on the 4th of October, 2003 when I was working under pressure to put finishing touches to the text of this inaugural lecture. As a friend and my immediate subordinate in the Housing Unit we were very close.

His greatest assets were his transparent honesty and fairness to others. May Allah grant him eternal rest.

Finally, I would like to bring this Inaugural Lecture to an end by what Paul said in 1 Corinthians 4[7]: "For who maketh thee to differ from another? and what hast thou that thou didst not receive? now if thou didst receive it, why dost thou glory, as if thou hadst not received it?"

Mr. Vice-Chancellor, Sir, distinguished ladies and gentlemen, I thank you for your patience and attention.
God bless.

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