# SYMBOLIC ANALYSIS OF HEAT TRANSFER IN RADIAL FIN

By

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***ABSTRACT***

*In this paper, a novel application domain of the symbolic implementation of finite element method using a linear heat dissipation radial fin has been presented. The problem of temperature distribution and heat transfer in radial fins of triangular, rectangular and parametric profiles was modeled and validated using symbolic computation. The current study has shown that the symbolic computational technique is less complex, effective and efficient in comparison with the earlier techniques used to solve heat transfer problem in this same problem domain. Our proposed concept could be adapted to solving heat transfer problems in further extended surfaces.*

***Keywords***: *Radial fin; finite element method; symbolic computation; Mathematica.*

## 1.0 INTRODUCTION

Numerical analysis is the usual recourse once boundary conditions and other factors preclude close-form solution to engineering problems. Finite Element Analysis (FEA) is perhaps the most successful approach to numerical computation of approximate solutions to such problems coupled with post-processing for simulation and sensitivity analysis. Commercial computational tools are widely available to implement several FEA schemes. Such canned programs often create a disconnection between the analyst and the problem as the whole process is rather mechanical. Computer Algebra System (CAS) is a program designed to perform symbolic and numerical manipulation following the rules of mathematics. Incorporating these with traditional FEA creates a middle ground where the development of the FEA schemes follows the same modeling approach as the symbolic representation of the underlying problems are directly accommodated.

Symbolic computation techniques performed by Computer Algebra System have found broad applications in many areas of science and engineering. It has led to new approaches for problems solving and provide tools that enable an automatic and computerized solution of problems in ways that are not possible with conventional computing systems. The number and quality of symbolic manipulation program has expanded dramatically since the availability of graphical workstation and personal computers have encouraged interactive and experimental programming, MATHEMATICA and MAPLE being the leading general purpose contenders, though there are more specialized programs.

Of importance in the study of computational techniques of heat transfer is the work of Campo et al (2008) who used mean value theorem in integration implementation, for solving the derived quasi-1D heat conduction equation. He used it to obtain an approximate analytical temperature distribution and heat transfer rates in annular fins of hyperbolic profiles while setting aside the conventional use of modified Bessel functions. He argued that the use of numerical evaluation with tools such as power series method, the finite-difference technique and the shooting method of temperature and heat transfer rates are extremely complicated even with symbolic codes.

Several of these numerical evaluation tools can be improved, sometimes considerably via embedded symbolic parts into the numerical algorithms. This is called hybrid techniques, because it involves numeric as well as symbolic manipulations. Jiang and Wang (2006) called this hybrid system "semi-symbolic program". They concluded that, "semi-symbolic program" written for the implementation of finite element method in plasticity is a good compromise between the computational efficiency and human effort in developing non-linear FEM program. In the paper, while developing the weak form of the governing differential equation, the shape function (or weight function), its derivatives, Jacobian and the strain-displacement matrix for each element are computed symbolically and stored in closed form. However, in order to maintain the equilibrium condition in Newton-Raphson iteration scheme, the evaluation of stiffness matrix and equivalent force of stress were performed numerically by using the Gauss-Legendre quadrature. This resulted in a semi-symbolic program of nonlinear finite element method.

A few other people have worked on the use of symbolic algebra system in the context of finite element method. Ioakimids (1993) solved an elastic problem to obtain a solution in terms of symbolic parameter. Yew et al (1995), Lee and Hobbs (1998) have obtained some closed form integration of the finite element stiffness matrix. Of the most important in the context of plasticity analysis with symbolics is the work of Korelc (2004). He developed a hybrid system in which MATHEMATICA was used for the automatic derivation of material model and the generation of symbolic nonlinear finite element codes. However, Korelc still relied on the transformation of MATHEMATICA codes into C codes for numerical evaluation. Papusha et al (2008) developed a symbolic solution to boundary value problems and applied it to solve problems in offshore design technology.

That means a good manipulation of symbolic implementation tool such as MATHEMATICA with in-built numerical evaluation schemes can help to achieve an entire implementation of FEA in symbolic form rather than semi symbolic or the hybrid system. The objective of this is study to develop a symbolic Finite Element solution to the radial heat transfer problem in fin of different profiles that can be used for design optimization. Such an approach is also very useful in the teaching of the FEA method since the symbolic solution paralleled the problem formulation directly.

## 2.0 FINITE ELEMENT FORMULATION FOR RADIAL FIN

The finite element method is an elementwise application of the variational method, in which a given differential equation is recast in an equivalent integral form (Reddy, 2006).

The finite element formulation for radial fin may be deduced from an energy balance for the steady-state condition of convective heat transfer and be reduced to equation (1) below:

$$\frac{d}{dr}\left(KA\frac{dT}{dr}\right)dr = 2\beta p(T - T_\infty)dr \qquad (1)$$

Where $\boldsymbol{K}$ is thermal conductivity, $\boldsymbol{\beta}$ is heat transfer coefficient, $\boldsymbol{A} = (2\pi r)2f$ (area normal to the heat flux), $\boldsymbol{P} = 2\pi r$ (perimeter of fin).

Therefore, the governing equation of heat transfer in radial fin is given by:

$$\frac{d}{dr}\left[rfk\frac{dT}{dr}\right]dr = \beta r(T - T_\infty)dr$$

Subject to $\quad T_{r=r_1} = T_b, \quad \frac{dT}{dr}|_{r=r_2} = 0 \qquad (2)$

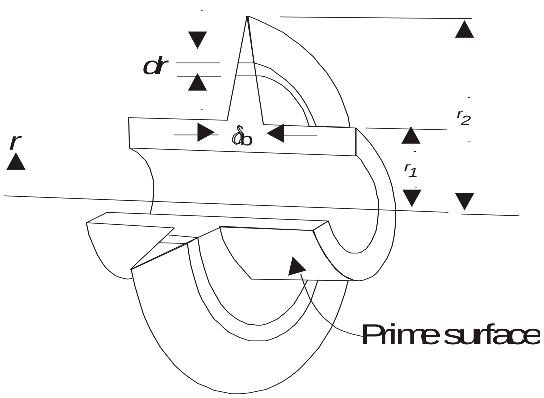



***Figure 1 Sketch of a radial fin of triangular profile***

Where $f$ is the shape function of a fin of arbitrary profile. For example, for a radial fin with triangular profile, $f = \frac{t(r_2 - r)}{2(r_2 - r_1)}$, and for rectangular profile $f = \frac{t}{2}$.

### 2.1 Weak Formulation

The weak form of equation (2) is obtained below by multiplying the equation with a weight function $w(x)$ and integrating over the domain $\Omega = (0, L)$ of the problem:

$$0 = \int_{r_1}^{r_2} w\left[\frac{d}{dr}\left(rfk\frac{dT}{dr}\right) - \beta r(T - T_\infty)\right]dr \qquad (3)$$

Integration by parts yields

$$0 = \int_{r_1}^{r_2}\left[rfk\frac{dw}{dr}\frac{dT}{dr} + \beta rw(T - T_\infty)\right]dr - w(r_1)\left(-rfk\frac{dT}{dr}\right)_{r_1} - w(r_2)\left(-rfk\frac{dT}{dr}\right)_{r_2} \qquad (4a)$$

$$0 = \int_{r_1}^{r_2}\left[rfk\frac{dw}{dr}\frac{dT}{dr} + \beta rwT\right]dr - \left[\int_{r_1}^{r_2}\beta rwT_\infty dr + w(r_1)Q_1 + w(r_2)Q_2\right] \qquad (4b)$$

To simplify further, we introduce the linear and bilinear forms

$$B(w,T) = \int_{r_1}^{r_2}\left[rfk\frac{dw}{dr}\frac{dT}{dr} + \beta rwT\right]dr \,,$$

$$l(w) = \left[\int_{r_1}^{r_2}\beta rwT_\infty dr + w(r_1)Q_1 + w(r_2)Q_2\right] (5)$$

In matrix notation, the linear algebraic equations can be written as

$$\left[K_{ij}^e\right]\{T^e\} = \{f_i^e\} \qquad (6)$$

Where

$$K_{ij}^e = \int_{r_1}^{r_2}\left[rfk\frac{d\psi_i^e}{dr}\frac{d\psi_j^e}{dr} + \beta r\psi_i^e\psi_j^e\right]dr \,,$$

$$f_i^e = \left[\int_{r_1}^{r_2}\beta rT_\infty\psi_i^e dr + \psi_i^e(r_1)Q_1 + \psi_i^e(r_2)Q_2\right] \qquad (7)$$

Where $\psi_1^e = \frac{r_2 - r}{h_e}$ and $\psi_2^e = \frac{r - r_1}{h_e}$ are the interpolation functions expressed in terms of radial coordinates for linear elements.

The symbolic forms of the element coefficients matrix $K_{ij}^e$ and the source vector $f_i^e$ are given below in radial coordinates;

$$K_{ij}^e = \begin{bmatrix} \frac{kt}{4} + \frac{h^2\beta}{12} + \frac{ktr_1}{2h} + \frac{1}{3}h\beta r_1 & -\frac{kt}{4} + \frac{h^2\beta}{12} - \frac{ktr_1}{2h} + \frac{1}{6}h\beta r_1 \\ -\frac{kt}{4} + \frac{h^2\beta}{12} - \frac{ktr_1}{2h} + \frac{1}{6}h\beta r_1 & \frac{kt}{4} + \frac{h^2\beta}{4} + \frac{ktr_1}{2h} + \frac{1}{3}h\beta r_1 \end{bmatrix} \qquad (8)$$

$$f_i^e = \begin{bmatrix} \frac{1}{6}h(h + 3\mathrm{r}1)\beta T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\mathrm{r}1\beta T_\infty \end{bmatrix} + \begin{bmatrix} \mathrm{Q}_1^\mathrm{e} \\ \mathrm{Q}_2^\mathrm{e} \end{bmatrix} \qquad (9)$$

### 2.2 Assembly of Elements into Master Matrix

In deriving the element equations, a typical element was isolated from the mesh and formulated the weak form and developed its finite element model. To obtain the finite element equation of the total problem, we must put the elements back into their original positions.

Using an 8-linear element node approach the symbolic form of the assembly is given below.

"Master Stiffness After Member Merge"

$$[K] = \begin{bmatrix} \frac{kt}{4} + \frac{h^2\beta}{12} + \frac{ktr_1}{2h} + \frac{1}{3}h\beta r_1 & -\frac{kt}{4} + \frac{h^2\beta}{12} - \frac{ktr_1}{2h} + \frac{1}{6}h\beta r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{kt}{4} + \frac{h^2\beta}{12} - \frac{ktr_1}{2h} + \frac{1}{6}hr_1 & \frac{kt}{4} + \frac{h^2\beta}{4} + \frac{ktr_1}{2h} + \frac{1}{3}h\beta r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10)$$

The matrix of the assembly of all the members (8 linear elements) is too large to be contained on the page of this paper.

"Master Source Vector Assembly of Elements"

$$[f] = \begin{bmatrix} \frac{1}{6}h\beta(h + 3r_1)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta r_1 T_\infty + \frac{1}{6}h\beta\big(h + 3(h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(h + r_1)T_\infty + \frac{1}{6}h\beta\big(h + 3(2h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(2h + r_1)T_\infty + \frac{1}{6}h\beta\big(h + 3(3h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(3h + r_1)T_\infty + \frac{1}{6}h\beta\big(h + 3(4h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(4h + r_1)T_\infty + \frac{1}{6}h\beta\big(h + 3(5h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(5h + r_1)T_\infty + \frac{1}{6}h\beta\big(h + 3(6h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(6h + r_1)T_\infty + \frac{1}{6}h\beta\big(h + 3(7h + r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty + \frac{1}{2}h\beta(7h + r_1)T_\infty \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 + Q_1^5 \\ Q_2^5 + Q_1^6 \\ Q_2^6 + Q_1^7 \\ Q_2^7 + Q_1^8 \\ Q_2^8 \end{bmatrix} \quad (11)$$

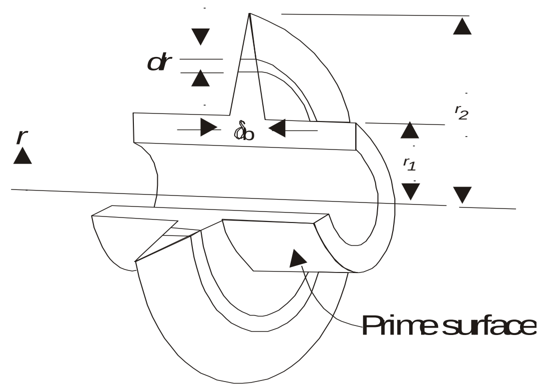
"Condensed Source Vector Assembly of Elements"

$$[f] = \begin{bmatrix} -T_b\left(-\frac{kt}{4}+\frac{h^2\beta}{12}-\frac{ktr_1}{2h}+\frac{1}{6}h\beta r_1\right)+\frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta r_1 T_\infty+\frac{1}{6}h\beta\big(h+3(h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(h+r_1)T_\infty+\frac{1}{6}h\beta\big(h+3(2h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(2h+r_1)T_\infty+\frac{1}{6}h\beta\big(h+3(3h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(3h+r_1)T_\infty+\frac{1}{6}h\beta\big(h+3(4h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(4h+r_1)T_\infty+\frac{1}{6}h\beta\big(h+3(5h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(5h+r_1)T_\infty+\frac{1}{6}h\beta\big(h+3(6h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(6h+r_1)T_\infty+\frac{1}{6}h\beta\big(h+3(7h+r_1)\big)T_\infty \\ \frac{1}{3}h^2\beta T_\infty+\frac{1}{2}h\beta(7h+r_1)T_\infty \end{bmatrix} \quad (12)$$

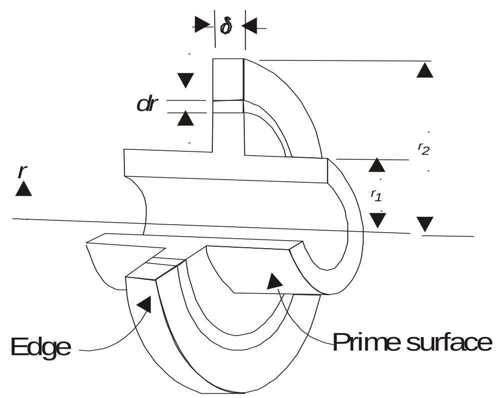
### 3.0 MODEL VALIDATION

The proposed finite element numerical scheme which is novel in its area of application in the current study is to be validated in the case study as presented below.

Consider a set of radial fins with triangular, rectangular and parametric profiles as shown in figure 2 below, and assuming the following parametric quantities hold for this radial fin:




(a)




(b)

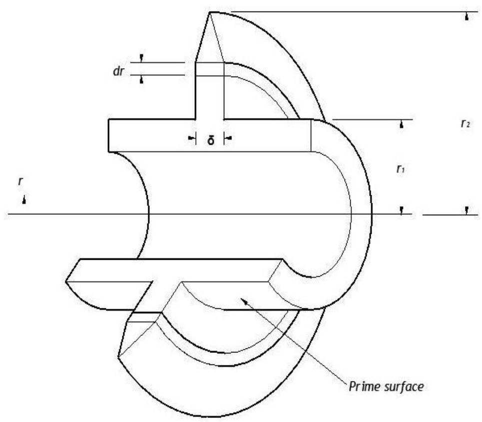



Figure 2 Radial fins (a)Triangular, (b)Rectangular, (c)Parametric Profiles

Inner and outer diameters are 10 and 25cm respectively, thickness of fin base is 0.25cm, the surrounding temperature is $35^0$C, while the temperature of fin base is $110^0$C,

If the thermal conductivity of the material which is made from steel is 40W/mK, and the heat transfer coefficient is 40W/m$^2$, the table and the graphs below give the summary of the results.

***(c) Figure 2 Radial fins (a) Triangular, (b) Rectangular, (c) Parametric Profiles***

***Table 1 Temperature distribution along radial fin with triangular, rectangular and parametric profiles***

| **TEMPERATURE DISTRIBUTION ALONG THE FIN** | | | |
|---|---|---|---|
| **NODE NO** | **TRIANGULAR $^0$C** | **RECTANGULAR $^0$C** | **PARAMETRIC $^0$C** |
| 1 | 110.0 | 110.0 | 110.0 |
| 2 | 91.0 | 89.0 | 89.8 |
| 3 | 76.7 | 74.6 | 76.1 |
| 4 | 65.8 | 64.7 | 67.0 |
| 5 | 57.4 | 58.0 | 61.0 |
| 6 | 51.1 | 53.4 | 53.7 |
| 7 | 46.4 | 50.6 | 48.2 |
| 8 | 42.9 | 49.0 | 44.2 |
| 9 (fin tip temp) | 41.0 | 48.5 | 42.0 |

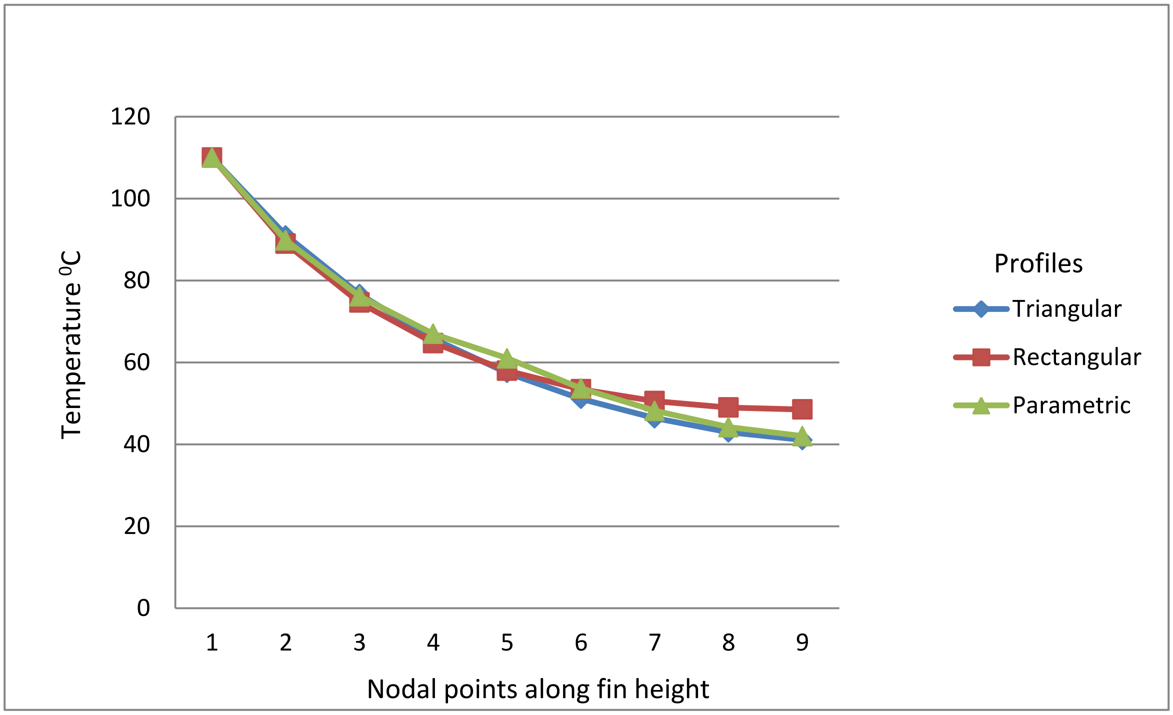



***Figure 3  Temperature distribution along fin height of different profiles.***

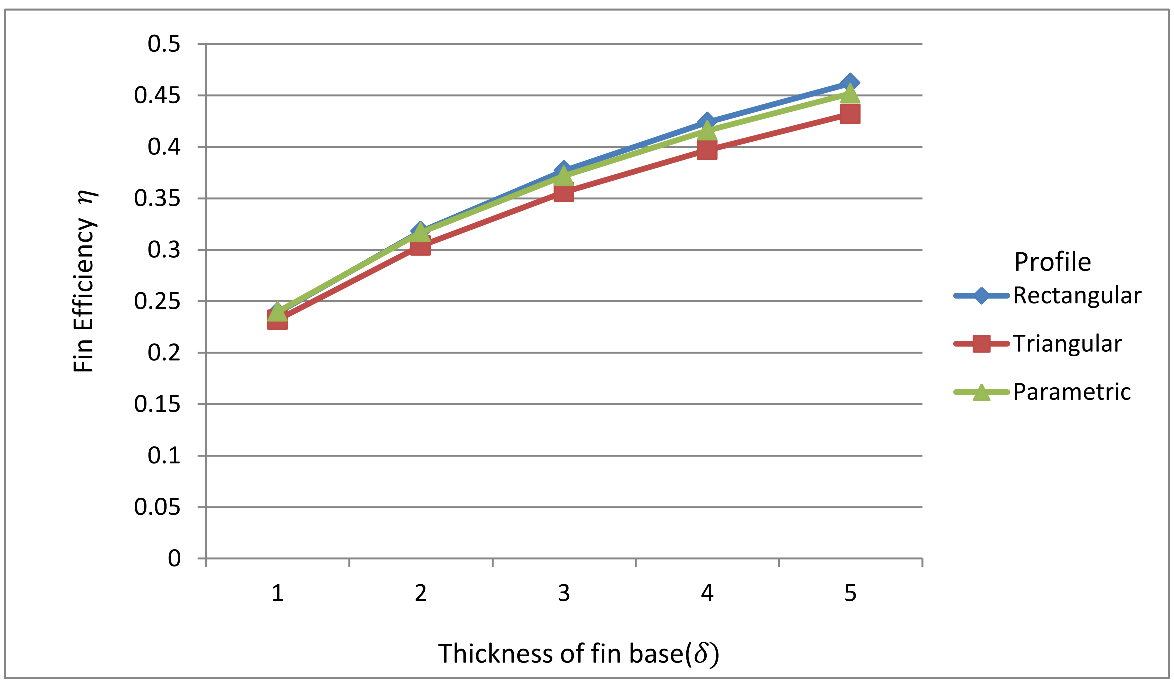



***Figure 4   Comparison of fin efficiencies: radial fins of rectangular, triangular and parametric profiles***

## RESULTS AND DISCUSSION

One of the key advantages of the Symbolic form of the Finite Element Analysis is that the entire solution is computed symbolically and stored in closed form. The Finite Element Analysis itself is a numerical method used in seeking approximate solution for practical problems that involve complicated domains (both geometry and material) where analytical solutions are not possible. Such is the problem of Radial fins with parametric profile.

The FEA analysis was carried out using the weak formulation approach also known as the Ritz method in the derivation of the element equation. In deriving the element equations, we isolated a typical element from the mesh, formulated the weak form and developed its finite element model. To obtain the finite element equation of the total problem, the elements were put back into their original positions (undoing of what was done before formulating the discrete problem).

In the discretization of the domain of the problem, we used eight (8) linear elements. The reason is not far-fetched. The finite element method is a powerful tool such that the higher the number of elements($n \rightarrow \infty$) used to discretize a domain, the faster the approximation solution converges to the exact solution. For instance, an infinite number of tiny line segments can approximate the perimeter of a circle. But this large number of elements invalidated computational efficiency in the result obtained from symbolic analysis. Therefore in order to achieve the main objective of this paper, and still have a solution that has almost zero error estimate, we used eight (8), linear elements.

In generating the condensed equation, the boundary conditions $T_{r=r_1} = T_b$ and $\frac{dT}{dr}|_{r=r_2} = 0$ which are of the mixed type were used. The boundary condition at $r_2$ indicates that heat in not dissipated at the tapered edge of the fin, since area of the edge is zero for triangular profile. For radial fins of other profiles, it was assumed that heat dissipation at $r_2$ is negligible.

The condensed equation can be applied directly in determining temperature distribution in radial fin of different profiles, since a general profile function has been built into the symbolic solution. In order to determine temperature distribution for specific profile, the profile function of the shape is substituted into the stiffness matrix or recoded into the program. For simplicity, the linear element was used and the Finite Element program was developed for a uniform mesh of arbitrary number of elements.

The performance of the fins is shown in the graphs below. Figure 3 shows the temperature distribution along the fin height in radial fin of different profiles, while Figure 4 shows the efficiencies of the fins with different profiles. From Figure 3, a minimal variation was observed in the temperature distribution along the fin height for these profiles. This minimal variation suggests that the fins will perform the same way regardless of the profile.

But a remarkable difference in their performance was observed when a normalization process was carried out. This was done by considering material usage of the fins. The material usage was gradually reduced as shown in Figure 5 below, where fin profile was changed from rectangular to triangular. The intermediate state is the parametric profile. At $\alpha = 0$, fin profile is completely rectangular, and then parametric at $\alpha$ ranging from 1 through 4 and finally triangular at $\alpha = 5$. This normalization is shown in Figure 6. In the figure, heat dissipated per material volume is plotted against $\alpha$, where $\alpha$ represents the gradual change of profile from rectangular to triangular.

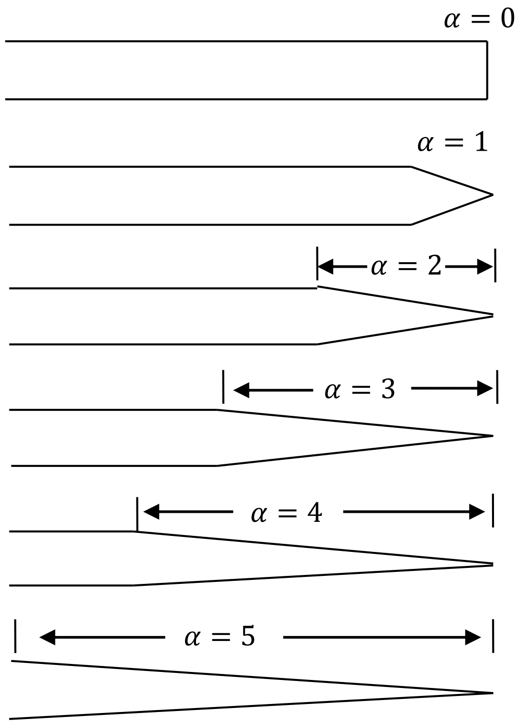



***Figure 5 Parametric Profile as the profile changes from rectangular to triangular.***

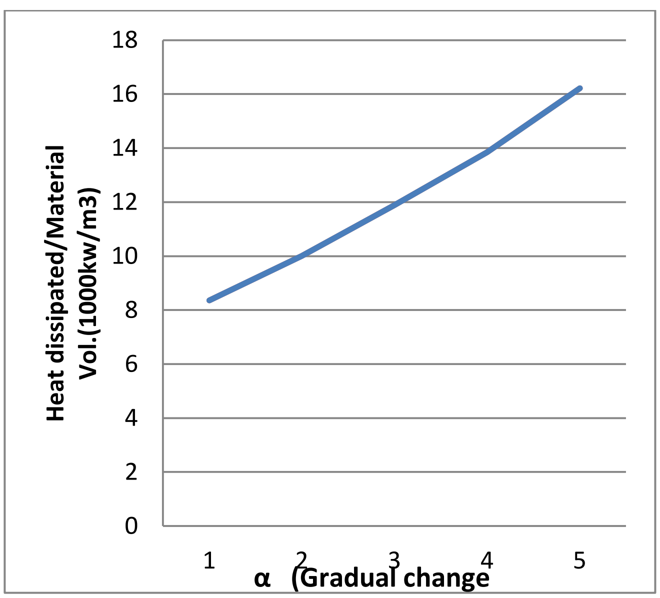



***Figure 6 Heat dissipated per material volume as profile changes from rectangular profile to triangular profile.***

This changing of fin profile or geometry from rectangular to triangular was done with ease in the symbolic computation. Doing it conventionally would require a significant increase in the complexity of mathematical manipulation. This is another advantage of symbolic computation. The result of symbolic computation as shown in the graph is useful for optimization of material usage. The graph shows that radial fin of triangular profile has a higher optimization of material usage in the design of radial fins than fins with rectangular profile. Considering the material usage, it dissipates heat by 90% more than radial fin with rectangular profile.

## CONCLUSION

It has been shown from this study that Finite Element Analysis of Radial Fin Heat Transfer can be implemented symbolically with Computer Algebra System (CAS) such as MATHEMATICA. It has also been shown that this symbolic computation technique is very effective and has the advantage to automatically derive the complex coefficients of the polynomial equations in matrix forms. It exempts one from using complex numerical methods to solve the polynomial equations. Modeling becomes very easy yet versatile.

The program for the implementation of the symbolic analysis of the radial fin is displayed in the appendix. The result of symbolic computation was used for optimization of material usage. This result showed that radial fin of triangular profile has a higher optimization of material usage in the design of radial fins than radial fin of rectangular profile.

**REFERENCES**


1. A. Campo, J. Cui, Temperature/Heat Analysis of Annular Fins of Hyperbolic Profile Relying on the Simple Theory for Straight Fins of Uniform Profile, Journal of Heat Transfer, May 2008, Vol. 130 / 054501- 1-4.

2. Y. Jiang and C. Wang, On teaching Finite Element Method in plasticity with MATHEMATICA, WILEY INTERSCIENCE, Computer Application in Engineering Education, December 2006, Vol. 16, Issue 3, Pages 233-242.

3. N. I. Ioakimids, Elementary application of MATHEMATICA to the solution of elasticity problems by the Finite Element Method, Computer Methods in Applied Mechanics and Engineering, 1993, Vol. 102, Pages 29-40.

4. C. K. Yew, J. T. Bole, and D. Mackenzie, Closed form integration of element stiffness matrices using a computer algebra system, SCIENCEDIRECT Computer and Structures, August 1995, Vol. 56, Issue 4, Pages 529-539.

5. C. K. Lee and R. E. Hobbs, Closed form stiffness matrix solution for some commonly used hybrid finite elements, SCIENCEDIRECT Computer and Structures, June 1998, Vol. 67, Issue 6, Pages 463-482.

6. J. Korelc, Symbolic formulation and automatic derivation of complex material models, Proceedings of Multi-Physics and Multi-Scale computer models in nonlinear analysis and optimal design of engineering structures under extreme conditions, Bled, Slovenia, 2004.

7. A. N. Papusha, I. V. Fedorov, V. V. Shtrasser, Symbolic evaluation of boundary problems for offshore design technology, The Mathematica Journal, 2008, Vol. 11, Issue 1.

8. J. N. Reddy, “An Introduction to the Finite Element Method”. Third Edition, 2006, pp 147 – 148, 164 – 168.