SOME PROBLEMS IN
INHOMOGENEOUS VISCOUS FLOW

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I CERTIFY THAT THIS THESIS 'SOME PROBLEMS IN INHOMOGENEOUS VISCOUS FLOW' IS THE RESULT OF THE WORK DONE BY THE CANDIDATE, MR. E. ADEJARE ADEBOYE UNDER MY SUPERVISION FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS OF THE UNIVERSITY OF LAGOS.

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ABSTRACT

This thesis serves a dual purpose. First, a method is presented which simplifies the integration of the governing steady state equations of an incompressible viscous fluid with vanishing body and inertia forces. The method takes advantage of potential theory to reduce any fluid problem to that of differential operators acting upon harmonic functions. This is in contrast to the conventional stream-function approach which employs a fourth-order partial differential equation. By this approach, the formulation for singularities in the interior of one of two immiscible fluids becomes straight forward.

Secondly, using this new approach, we are able to show in two selected examples that from a previously determined solution for a fluid occupying the entire three-dimensional space we can deduce the velocities and stress components for the case in which the entire three-dimensional space is occupied by two immiscible dissimilar fluids with a spherical interface. The selected examples are those in which the flow field in the homogeneous fluid can be described by a harmonic function namely, the case of a source in an unbounded fluid space (an axisymmetric flow problem) and the case of shear flow (which is non-axisymmetric). Also in chapter three an attempt is made to develop a theory for the flow field of a thin jet in an unbounded viscous fluid.

CONTENTS

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CHAPTER ONE

1.1 Introduction

A lot of work had been done on the flow fields of an inviscid fluid. Attempts have been made to compare the flow field of the disturbed stream with the flow field in the absence of obstacles. In 1940, Milne-Thomson (18) put forward his famous circle theorem viz: in a two dimensional irrotational flow of an incompressible inviscid fluid in the z-plane, with no rigid boundaries, if the complex potential is f(z) (the singularities of f(z) being all at a distance greater than 'a' from the origin), then on introducing a circular cylinder $|z| = a$ into the field the complex potential becomes

$$f(z) + \bar{f}(a^2/z)$$

In 1954, E. Levin extended the circle theorem to include the case in which rigid boundaries are present. In 1944, Weiss (29) considered the case when the obstacle is a sphere and he gave a theorem that if there be irrotational flow, of incompressible inviscid fluid with no rigid boundaries, characterised by velocity potential $\phi(r,\theta,\omega)$, all of whose singularities are at a distance greater than 'a' from the origin, then the introduction of a sphere $r = a$ into the field changes the velocity potential to

$$\phi(r,\theta,\omega) + \frac{1}{a}\int_0^{a^2/r} R\frac{\partial}{\partial R}\,\phi(R,\theta,\omega)\,dR$$

where $R = a^2/r$.

Then in 1953, Butler (7) considered the same problem using the stream function instead of the velocity potential and showed that if the original stream function in an axisymmetric irrotational flow is $\psi_0(r,\theta)$, then the introduction of a sphere $r = a$ changes the stream function to $\psi_0(r,\theta) - \frac{r}{a}\psi_0(a^2/r,\theta)$.

The results of Mile-Thomson, Levin, Weiss and Butler in a nutshell can be stated that the flow field in the presence of a cylinder or a sphere can be written directly from the known field without obstacles in the case of homogeneous inviscid fluid. Unfortunately, no fluid is perfectly inviscid so that these theorems are mainly of theoretical interest.

In a viscous fluid flow, most of the work done is centered round the case when the obstacle is a flat plate particularly after Prandtl (21) who in 1904 gave the theory of the boundary layer. The works in 1961 of Roshko (22), in 1963 of Bloor (6), Catheral and Mangler (8); in 1964 of Pearson (20) and Srivastava (24) and in 1969 of Talke and Berger (25) are a few typical efforts based mainly on the boundary layer theorem.

In 1906 and 1911, Einstein (9) did some work on the viscosity of a fluid in which small solid spheres are suspended and his work was extended by Jefferys (12) in 1922. Attempt was made by Adeboye (1) in dealing with flow in the presence of obstacles of regular shapes, (solids with circular, elliptical and parabolic cross-sections), involving fluids of low viscosity.

Not much work has so far been done on problems involving two-phase flows, that is when the solid obstacles are replaced by fluid ones. This is because, according to Taylor (26), "The difficulties in the way of a complete theory when solid particles are replaced by fluid drops are almost insuperable, partly because the correct boundary conditions are not known and partly because a fluid drop would deform under the combined action of viscous forces and surface tension."

This opinion is shared by Aderogba (2) who observes, "The problem presented by the state of stress in two immiscible fluids under a general internal load is a very complicated one not only because of the non-linear nature of the governing Navier-Stoke's equations but also because of the ever-changing interface conditions".

However, Taylor observed that a fairly accurate theory could be developed if it is assumed that the radius of the suspended drops or the velocity of distortion of the fluid are small, since in that case surface tension may be expected to keep the drops nearly spherical. Concerning the boundary conditions also, Aderogba affirmed that in some cases a combination of facts and methods available in the bending and stretching of bonded materials (in elasticity) can yield significant research results in fluid dynamics.

Therefore, while Taylor (27) tried experimentally to calculate the distortion of a drop of one fluid by viscous forces associated with certain Mathematically definable fields of flow of another fluid which surrounds it and Wohl and Rubinov (23) studied the movement of a deformable liquid sphere in an unbounded steady parabolic flow with the aim of determining the transverse force on the drop, it is the attempt in (2) and (3) which has direct bearings on our present studies.

In the paper (2) the case when the entire three-dimensional space is filled with two incompressible immiscible viscous fluids with a circular, cylindrical interface has been considered. It is shown that when a line source is located in one of the fluids, then the velocities and stresses produced in the two-fluid space can be obtained by differentiation of velocities and stresses which would be produced in a single-fluid space.

In (3) it is shown that if the displacement in the homogeneous infinite plane is the gradient of a harmonic function, then the displacement in a bonded circular disc is the gradient of a scalar multiple of the same harmonic function.

1.2. Aims and Method

The objects of this thesis are mainly to investigate an extension of the work in (2) to the case when the interface between the two fluids is spherical, that is, we would consider the case when the entire three-dimensional space is filled with two incompressible immiscible viscous fluids with a spherical interface. We would then investigate what relationship, if any, exists between the velocities and stresses in the two-fluid space and a single-fluid space when

(a) the flow is axisymmetric,

(b) the flow is not axisymmetric.

The Stokes' stream function is capable of solving only three-dimensional problems which are axisymmetric. In Chapter two we attempt to develop a new method of solving the Navier-Stoke's equations for cases where the flow is more general in nature, that is, axisymmetric or otherwise. Towards the end of the chapter we demonstrate how this method can be used in solving in an elegant manner the problem of a solid sphere in a viscous fluid in uniform flow.

In chapter three, we apply the same method to a more difficult problem. We attempt to develop a theory for the flow of a thin jet in an infinite viscous fluid.

In chapter four, we consider the case when the entire three-dimensional space is filled with two immiscible viscous fluids with a spherical interface, one fluid in the region $r > a$ the other in the region $r < a$, while a point source is located in the region $r > a$. We assume that 'a' is sufficiently small, so that surface tension alone could keep the drop spherical; that the velocities and tangential stresses are continuous at the interface and that body forces and inertial terms in the Navier-Stoke's equations are negligible. We also consider the particular case when the region $r < a$ is occupied by a spherical bubble of gas of negligible viscosity.

In chapter five, we consider a non-axisymmetric flow problem in which the entire three-dimensional space is filled with a viscous fluid undergoing shear flow in the presence of a spherical drop of a dissimilar viscous fluid occupying the region $r < a$. We use the same boundary conditions as in chapter four and employ the method of solution established in chapter two. The case when the spherical drop contains a fluid whose viscosity is much less than that of the surrounding fluid is also inferred.

Chapter six contains a summary of our conclusions and references.

CHAPTER TWO

A general solution of the Navier-Stoke's equation for steady-state incompressible fluid flow

As we observed earlier on in chapter one, the Stokes' stream function can only be applied to three-dimensional flow problems which are axisymmetric. Now we intend to investigate flow problems which may be axisymmetric or otherwise and for the sake of continuity we would like to use the same method throughout. In this manner we develop a general solution of the Navier-Stoke's equation which can be successfully applied in solving problems axisymmetric or otherwise.

Solving the Navier-Stoke's equation for steady-state incompressible fluid flow when the body forces and inertial terms are negligible normally involves the solution of a biharmonic equation

$$\nabla^4 \psi = 0,$$

where ψ represents the stream function. Our new approach replaces this biharmonic equation with a system of harmonic equations

$$\nabla^2 \phi_0 = 0, \; \nabla^2 \underset{\sim}{\phi} = \underset{\sim}{0}$$

where $\underset{\sim}{\phi} = (\phi_1, \phi_2)$ in two dimensions

$= (\phi_1, \phi_2, \phi_3)$ in three dimensions.

The solution obtained by this method gives the velocity vector and pressure at a point as

$$\underset{\sim}{q} = \nabla(\phi_0 + \underset{\sim}{r} \cdot \underset{\sim}{\phi}) - 2\,\underset{\sim}{\phi},$$

$$p = 2\mu(\nabla \cdot \underset{\sim}{\phi}).$$

2.1 Biharmonic aspect of the problem:

The steady-state Navier-Stoke's equation appropriate to an incompressible viscous fluid, in the absence of body forces and inertial terms is

$$\nabla p = \mu \nabla^2 \underset{\sim}{q} \tag{1}$$

together with the equation of continuity

$$\nabla . \underset{\sim}{q} = 0 \tag{2}$$

where p is the pressure, $\underset{\sim}{q}$ is the velocity vector and μ is the coefficient of viscosity.

Equation (1) is equivalent, in two-dimensions, to the system of equations

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q_x \qquad &(a) \\ \frac{\partial p}{\partial y} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q_y \qquad &(b) \end{aligned}\right\} \tag{3}$$

Differentiating (3a) partially with respect to y and (3b) partially with respect to x and subtracting,

$$0 = \mu\left(\frac{\partial^3 q_x}{\partial y \partial x^2} + \frac{\partial^3 q_x}{\partial y^3} - \frac{\partial^3 q_y}{\partial x^3} - \frac{\partial^3 q_y}{\partial x \partial y^2}\right) \tag{4}$$

If $q_x = \frac{\partial \psi}{\partial y}$, $q_y = -\frac{\partial \psi}{\partial x}$, then (4) passes over into

$$0 = \mu\left(\frac{2\partial^4 \psi}{\partial y^2 \partial x^2} + \frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial x^4}\right)$$

i.e. $$\nabla^4 \psi = 0 \tag{5}$$

Equation (5) is biharmonic, so that solving equation (1) involves solving a biharmonic equation.

We now propose a new approach whereby the solution of (1) could be achieved by solving a system of harmonic equations instead of a biharmonic one.

2.2 Solution using harmonics:

Suppose

$$\underset{\sim}{q} = \nabla\phi_0 \tag{6}$$

Then (6) satisfies the continuity equations (2) i.e. $\nabla . \underset{\sim}{q} = 0$ provided that $\nabla^2\phi_0 = 0$.

Substituting (6) into (1) we get

$$\nabla p = \mu \nabla (\nabla^2\phi_0) = 0 , \quad \text{if} \quad \nabla^2\phi_0 = 0$$

$$\Rightarrow p = \text{constant.}$$

$$= 0, \text{ without loss of generality,}$$

$\therefore$ $p = 0$ and $\underset{\sim}{q} = \nabla\phi_0$ constitute a solution of (1) provided $\nabla^2\phi_0 = 0$.

Now let

$$q_x = y\frac{\partial\phi_2}{\partial x} , \qquad q_y = y\frac{\partial\phi_2}{\partial y} - \phi_2 \tag{7}$$

substituting (7) into (3a) we obtain

$$\frac{\partial p}{\partial x} = \mu\left[y\frac{\partial}{\partial x}\left(\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2}\right) + 2\frac{\partial^2\phi_2}{\partial x\partial y}\right]$$

$$= 2\mu\frac{\partial^2\phi_2}{\partial x\partial y} \quad \text{if} \quad \nabla^2\phi_2 = 0$$

$\therefore$ i.e. $p = 2\mu \frac{\partial \phi_2}{\partial y} + f_1(y)$

A substitution of (7) into (3b) yields

$$p = 2\mu \frac{\partial \phi_2}{\partial y} + f_2(x)$$

$$\therefore \quad p = 2\mu \frac{\partial \phi_2}{\partial y}$$

$$\therefore \quad q_x = y \frac{\partial \phi_2}{\partial x}, \quad q_y = y \frac{\partial \phi_2}{\partial y} - \phi_2 , \; p = 2\mu \frac{\partial \phi_2}{\partial y} \tag{8}$$

constitute a solution of (3) or (1) provided $\nabla^2 \phi_2 = 0$.

Similarly

$$q_x = x \frac{\partial \phi_1}{\partial x} - \phi_1, \quad q_y = x \frac{\partial \phi_1}{\partial y}, \quad p = 2\mu \frac{\partial \phi_1}{\partial x} \tag{9}$$

constitute a solution of (3) or (1) provided $\nabla^2 \phi_1 = 0$.

Now since the equations are linear, the sum of the solutions is also a solution and hence from (8) and (9)

$$\begin{aligned} q_x &= x \frac{\partial \phi_1}{\partial x} + y \frac{\partial \phi_2}{\partial x} - \phi_1 \\ q_y &= x \frac{\partial \phi_1}{\partial y} + y \frac{\partial \phi_2}{\partial y} - \phi_2 \\ p &= 2\mu \left(\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} \right) \end{aligned} \tag{10}$$

constitute a solution of (3) or (1) provided $\nabla^2 \phi_1 = 0 = \nabla^2 \phi_2$.

Now in the case of three-dimensions equation (1) is equivalent to

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) q_x \\
\frac{\partial p}{\partial y} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) q_y \qquad (11) \\
\frac{\partial p}{\partial z} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) q_z
\end{aligned}$$

By a procedure similar to the one given above, equation (11) is found to be satisfied by

$$p = 2\mu\left(\frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial y} + \frac{\partial\phi_3}{\partial z}\right)$$

$$q_x = x\frac{\partial\phi_1}{\partial x} + y\frac{\partial\phi_2}{\partial x} + z\frac{\partial\phi_3}{\partial x} - \phi_1$$

$$q_y = x\frac{\partial\phi_1}{\partial y} + y\frac{\partial\phi_2}{\partial y} + z\frac{\partial\phi_3}{\partial y} - \phi_2$$

$$q_z = x\frac{\partial\phi_1}{\partial z} + y\frac{\partial\phi_2}{\partial z} + z\frac{\partial\phi_3}{\partial z} - \phi_3$$

This can be verified by a direct substitution of (12) into (11).

Therefore (12) constitute a solution of (1) in three-dimensions.

Now q_x and q_y as given in (10) and q_x, q_y and q_z as given in (12) are the components of $\nabla(\underset{\sim}{r}.\underset{\sim}{\phi}) - 2\underset{\sim}{\phi}$ in two-and three-dimensions respectively where

$\underset{\sim}{\phi} = (\phi_1, \phi_2)$ in two-dimensions

$\quad = (\phi_1, \phi_2, \phi_3)$ in three-dimensions.

Also p can be written as

$$p = 2\mu(\nabla . \underset{\sim}{\phi})$$

Therefore, as a check, we put

$$\underset{\sim}{q} = \nabla(\underset{\sim}{r} . \underset{\sim}{\phi}) - 2\underset{\sim}{\phi} \qquad (13)$$

and substitute in (1) .

Now (1) can be written as

$$\nabla p = \mu[\nabla(\nabla . \underset{\sim}{q})] - \nabla_{\wedge}(\nabla_{\wedge}\underset{\sim}{q})]$$

$$= - \mu[\nabla_{\wedge}(\nabla_{\wedge}\underset{\sim}{q})] \qquad (14)$$

since $\nabla . \underset{\sim}{q} = 0$, for incompressible fluid flow.

Therefore substituting (13) into (14) we get

$$\nabla p = - \mu \left\{\nabla_{\wedge}[\nabla_{\wedge}\nabla(\underset{\sim}{r} . \underset{\sim}{\phi})]\right\} + 2\mu[\nabla_{\wedge}(\nabla_{\wedge}\underset{\sim}{\phi})]$$

$$= 2\mu[\nabla_{\wedge}(\nabla_{\wedge}\underset{\sim}{\phi})], \quad \text{since } \nabla_{\wedge}[\nabla(\underset{\sim}{r} . \underset{\sim}{\phi})] = \underset{\sim}{0}$$

$$= 2\mu[\nabla(\nabla . \underset{\sim}{\phi}) - \nabla^2 \underset{\sim}{\phi}]$$

$$= 2\mu\nabla(\nabla . \underset{\sim}{\phi}), \text{ if } \nabla^2 \underset{\sim}{\phi} = \underset{\sim}{0}$$

$\Rightarrow p = 2\mu(\nabla . \underset{\sim}{\phi})$, taking the constant of integration to be zero. Therefore equation (14) or (1) is indeed satisfied by

$$p = 2\mu(\nabla . \underset{\sim}{\phi}), \quad \underset{\sim}{q} = \nabla(\underset{\sim}{r} . \underset{\sim}{\phi}) - 2\underset{\sim}{\phi} \qquad (15)$$

provided $\nabla^2 \underset{\sim}{\phi} = \underset{\sim}{0}$.

Consequently we have two solutions of (1) given by (6) and (15). Therefore a general solution of (1) can be written as their sums, i.e.

$$
\begin{aligned}
p &= 2\mu\nabla.\underset{\sim}{\phi} \\
\underset{\sim}{q} &= \nabla\phi_0 + \nabla(\underset{\sim}{r}.\underset{\sim}{\phi}) - 2\underset{\sim}{\phi} \\
&= \nabla(\phi_0 + \underset{\sim}{r}.\underset{\sim}{\phi}) - 2\underset{\sim}{\phi}
\end{aligned} \tag{16}
$$

where ϕ_0 and $\underset{\sim}{\phi}$ satisfy the harmonic equations

$$\nabla^2\phi_0 = 0 \text{ and } \nabla^2\underset{\sim}{\phi} = \underset{\sim}{0}.$$

2.3 Observation. We observe from (16) that for a general solution of the Navier-Stoke's equation in steady state, when the fluid is viscous and incompressible and in the absence of body forces and inertial terms, we need to solve <u>at most three</u> harmonic equations for any two-dimensional flow problem and <u>at most four</u> harmonic equations for any three-dimensional flow problem.

2.4 Applications.

(a) A rigid sphere in an infinite fluid. As a check on the solution constructed above, consider a flow which at infinity has a velocity vector given by

$$\underset{\sim}{q}^{\infty} = (0,0,-U) \tag{17}$$

Let a rigid sphere of radius 'a' be introduced into the flow with its centre at the origin. The motion in this case would be axisymmetric. We shall now construct the flow field of the disturbed stream.

The solution of the equation

$$\nabla p = \mu \nabla^2 \underset{\sim}{q}$$

is, as shown above,

$$\underset{\sim}{q} = \nabla(\phi_0 + \underset{\sim}{r}\cdot\underset{\sim}{\phi}) - 2\underset{\sim}{\phi}$$

$$p = 2\mu(\nabla\cdot\underset{\sim}{\phi})$$

where $\underset{\sim}{\phi} = (\phi_1, \phi_2, \phi_3)$ and ϕ_0 are the solutions of the harmonic equations

$$\nabla^2\phi_0 = 0 \quad \text{and} \quad \nabla^2\underset{\sim}{\phi} = \underset{\sim}{0}$$

Now in spherical coordinate system,

$$\nabla = \underset{\sim}{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\underset{\sim}{e}_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin\theta}\underset{\sim}{e}_\varphi \frac{\partial}{\partial \varphi} \tag{18}$$

and

$$\begin{aligned}\underset{\sim}{\phi} = {} & (\phi_1 \sin\theta\cos\varphi + \phi_2 \sin\theta\sin\varphi + \phi_3 \cos\theta)\underset{\sim}{e}_r \\ & + (\phi_1 \cos\theta\cos\varphi + \phi_2 \cos\theta\sin\varphi - \phi_3 \sin\theta)\underset{\sim}{e}_\theta \\ & + (\phi_2 \cos\varphi - \phi_1 \sin\varphi)\underset{\sim}{e}_\varphi \end{aligned} \tag{19}$$

while

$$\underset{\sim}{r} = r\underset{\sim}{e}_r \tag{20}$$

From (16), (18), (19) and (20)

$$q_r = \frac{\partial}{\partial r}[\phi_0 + r \sin\theta(\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 r \cos\theta] - 2[\sin\theta(\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 \cos\theta] \tag{21}$$

$$q_\theta = \frac{\partial}{r\partial\theta}[\phi_0 + r \sin\theta(\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 r \cos\theta] - 2[\cos\theta(\phi_1 \cos\varphi + \phi_2 \sin\varphi) - \phi_3 \sin\theta] \tag{22}$$

$$q_\varphi = \frac{1}{r \sin\theta} \frac{\partial}{\partial\varphi}[\phi_0 + r \sin\theta(\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 r \cos\theta] - 2(\phi_2 \cos\varphi - \phi_1 \sin\varphi) \tag{23}$$

Now, since the motion under consideration is axisymmetric,

$$q_\varphi \equiv 0 \quad \text{for all } \varphi .$$

Therefore from (23), we see that for q_φ to be zero for all φ, we must set

$$\phi_1 = \phi_2 = 0 \tag{24}$$

while ϕ_0 and ϕ_3 must be independent of φ.

Then from (21) and (22), q_r and q_θ reduce respectively to

$$q_r = \frac{\partial}{\partial r}(\phi_0 + \phi_3 r \cos\theta) - 2\phi_3 \cos\theta \tag{25}$$

$$q_\theta = \frac{\partial}{r\partial\theta}(\phi_0 + \phi_3 r \cos\theta) + 2\phi_3 \sin\theta \tag{26}$$

Consequently, we need to consider only ϕ_0 and ϕ_3 in this case.

In spherical coordinates, $\underset{\sim}{q}^\infty$ as given by (17) is

$$\underset{\sim}{q}^\infty = -U \cos\theta\, \underset{\sim}{e}_r + U \sin\theta\, \underset{\sim}{e}_\theta \tag{27}$$

So that

$$q_r^{\infty} = - U \cos \theta \ , \quad q_\theta^{\infty} = U \sin \theta \tag{28}$$

Following the introduction of the rigid sphere, the boundary conditions to be satisfied on the sphere are

$$q_r^{(1)} = q_\theta^{(1)} = 0 \quad \text{at} \quad r = a \tag{29}$$

where

$$q_r^{(1)} = q_r^{\infty} + q_r^{*}, \quad q_\theta^{(1)} = q_\theta^{\infty} + q_\theta^{*} \tag{30}$$

so that q_r^{*} and q_θ^{*} are the perturbations in radial and tangential components of velocity.

We now need to solve for values of ϕ_0 and ϕ_3 in the equations

$$\nabla^2 \phi_0 = 0, \quad \nabla^2 \phi_3 = 0 \tag{31}$$

such that the conditions (29) are satisfied.

From (28), we observe the dependence of $q_r^{(1)}$ on $\cos \theta$ and the dependence of $q_\theta^{(1)}$ on $\sin \theta$. Furthermore, (29) suggest that both depend on r. Guided by these facts, and observing the nature of (25) and (26), we construct a solution of (31) in the form

$$\begin{aligned} \phi_0 &= g(r) \cos \theta \\ \phi_3 &= f(r) \end{aligned} \tag{32}$$

where $g(r)$ and $f(r)$ are functions of r only.

Now in spherical polar coordinates and for axisymmetric flows,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} \tag{33}$$

Hence $\nabla^2 \phi_0 = 0$ becomes on using (32) and (33),

$$\cos\theta\left(\frac{d^2 g}{dr^2} + \frac{2}{r}\frac{dg}{dr} - \frac{2}{r^2}g\right) = 0 \tag{34}$$

A solution of (34) is

$$g = \frac{A}{r^2} \tag{35}$$

where A is a constant.

The other solution, $g = kr$, k a constant, is inadimissible since the effect of the introduction of the sphere must tend to zero as $r \to \infty$.

Also from (32) and (33), $\nabla^2 \phi_3 = 0$ becomes

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0 \tag{36}$$

The solution of (36) is

$$f = \frac{B}{r} + C \tag{37}$$

where B and C are constants. However, $C = 0$ in this case since the effect of the disturbance must vanish as $r \to \infty$.

$$\therefore \quad \begin{aligned} \phi_0 &= \frac{A}{r^2}\cos\theta \\ \phi_3 &= \frac{B}{r} \end{aligned} \tag{38}$$

Hence from (25) and (26)

$$q_r^* = \frac{\partial}{\partial r}\left(\frac{A}{r^2} + B\right)\cos\theta - \frac{2B}{r}\cos\theta$$

$$= -2\cos\theta\left(\frac{A}{r^3} + \frac{B}{r}\right) \qquad (39)$$

$$q_\theta^* = \frac{\partial}{r\partial\theta}\left[\left(\frac{A}{r^2} + B\right)\cos\theta\right] + \frac{2B}{r}\sin\theta$$

$$= -\sin\theta\left(\frac{A}{r^3} - \frac{B}{r}\right) \qquad (40)$$

So that from (28), (30), (39) and (40),

$$q_r^{(1)} = -\cos\theta\left(\frac{2A}{r^3} + \frac{2B}{r} + U\right) \qquad (41)$$

$$q_\theta^{(1)} = -\sin\theta\left(\frac{A}{r^3} - \frac{B}{r} - U\right) \qquad (42)$$

So that when $r = a$,

$$q_r^{(1)} = -(\cos\theta)\left(\frac{2A}{a^3} + \frac{2B}{a} + U\right) = 0 \text{ from } (29)$$

or

$$\frac{2A}{a^3} + \frac{2B}{a} = -U \qquad (43)$$

Also from (29) and (42), when $r = a$,

$$q_\theta^{(1)} = -(\sin\theta)\left(\frac{A}{a^3} - \frac{B}{a} - U\right) = 0$$

or

$$\frac{A}{a^3} - \frac{B}{a} = U \tag{44}$$

Solving (43) and (44) simultaneously gives

$$A = \frac{a^3 U}{4}, \qquad B = -\frac{3aU}{4} \tag{45}$$

Hence

$$q_r^{(1)} = \left(-\frac{2a^3U}{4r^3} + \frac{6aU}{4r} - U\right)\cos\theta$$

$$= -U\cos\theta - \tfrac{1}{2}U\left(\frac{a^3}{r^3} - \frac{3a}{r}\right) \tag{46}$$

and

$$q_\theta = U\sin\theta - \frac{1}{4}U\left(\frac{a^3}{r^3} + \frac{3a}{r}\right)\sin\theta \tag{47}$$

which agrees with results obtained by traditional methods.

(b) Stress components. The stress components are given in spherical coordinates as

$$\sigma_{rr} = -p + 2\mu\frac{\partial q_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r}\left(\frac{\partial q_\theta}{\partial\theta} + q_r\right)$$

$$\sigma_{\varphi\varphi} = -p + \frac{2\mu}{r\sin\theta}\left(\frac{\partial q_\varphi}{\partial\varphi} + q_r\sin\theta + q_\theta\cos\theta\right) \tag{48}$$

$$\sigma_{r\theta} = \mu\left(\frac{\partial q_r}{r\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right)$$

$$\sigma_{r\varphi} = \mu\left(\frac{\partial q_r}{r\sin\theta\ \theta\theta} + \frac{\partial q_\varphi}{\partial r} - \frac{q_\varphi}{r}\right)$$

$$\sigma_{\theta\varphi} = \mu\left(\frac{\partial q_\varphi}{r\partial\theta} + \frac{\partial q_\theta}{r\sin\theta\ \theta\varphi} - \frac{q_\varphi\cot\theta}{r}\right)$$

where

$$p = \frac{2\mu}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(r^2\sin\theta\ q_r) + \frac{\partial}{\partial\theta}(r\sin\theta\ q_\theta) + \frac{\partial}{\partial\varphi}(rq_\varphi)\right)$$

In term of the harmonic functions ϕ_0, ϕ_1, ϕ_2, and ϕ_3 therefore, if we let

$$\lambda = \phi_0 + r\sin\theta(\phi_1\cos\varphi + \phi_2\sin\varphi) + \phi_3\cos\theta \tag{49}$$

we can write, from (21), (22) and (23)

$$p = \frac{2\mu}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left[r^2\sin\theta\left(\frac{\partial\lambda}{\partial r} - 2[\sin\theta\ (\phi_1\cos\varphi + \phi_2\sin\varphi)\right.\right.\right.$$

$$\left.\left. + \phi_3\cos\theta]\right)\right] + \frac{\partial}{\partial\theta}\left[r\sin\theta\left(\frac{\partial\lambda}{r\partial\theta}\right.\right.$$

$$\left.\left. - 2[\cos\theta\ (\phi_1\cos\varphi + \phi_2\sin\varphi) - \phi_3\sin\theta]\right)\right]$$

$$\left. + \frac{\partial}{\partial\varphi}\left[\frac{\partial\lambda}{\sin\theta\ \theta\varphi} - 2r(\phi_2\cos\varphi - \phi_1\sin\varphi)\right]\right\} \tag{50}$$

$$\sigma_{rr} = -p + 2\mu\left\{\frac{\partial^2\lambda}{\partial r^2} - 2\,\frac{\partial}{\partial r}[\sin\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi) + \phi_3\cos\theta]\right\} \tag{51}$$

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r}\left\{\frac{\partial^2\lambda}{r\partial\theta^2} - 2\,\frac{\partial}{\partial\theta}[\cos\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi) - \phi_3\sin\theta] + \frac{\partial\lambda}{\partial r} - 2[\sin\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi) + \phi_3\cos\theta]\right\} \tag{52}$$

$$\sigma_{\varphi\varphi} = -p + \frac{2\mu}{r\sin\theta}\left\{\frac{1}{r\sin\theta}\,\frac{\partial^2\lambda}{\partial\varphi^2} - 2\frac{\partial}{\partial\varphi}(\phi_2\cos\varphi - \phi_1\sin\varphi) + \sin\theta\left(\frac{\partial\lambda}{\partial r} - 2[\sin\theta\,(\phi_1\cos\varphi+\phi_2\sin\varphi) + \phi_3\cos\theta]\right) + \cos\theta\left(\frac{\partial\lambda}{r\partial\theta} - 2[\cos\theta\,(\phi_1\cos\varphi+\phi_2\sin\varphi) - \phi_3\sin\theta]\right)\right\} \tag{53}$$

$$\sigma_{r\theta} = \mu\left\{\frac{2\partial^2\lambda}{r\partial r\partial\theta} - \frac{2}{r}\,\frac{\partial}{\partial\theta}[\sin\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi) + \phi_3\cos\theta] - 2\,\frac{\partial}{\partial r}[\cos\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi) - \phi_3\sin\theta] - \frac{\partial\lambda}{r^2\partial\theta} + \frac{2}{r}[\cos\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi) - \phi_3\sin\theta]\right\} \tag{54}$$

$$\sigma_{r\varphi} = \mu\left\{\frac{1}{r\sin\theta}\left(\frac{2\partial^2\lambda}{\partial r\partial\varphi} - 2\,\frac{\partial}{\partial\varphi}[\sin\theta\,(\phi_1\cos\varphi + \phi_2\sin\varphi)\right.\right.$$

$$+ \phi_3 \cos\theta]\Big) - 2\frac{\partial}{\partial r}(\phi_2 \cos\varphi - \phi_1 \sin\varphi)$$

$$- \frac{1}{r^2 \sin\theta}\frac{\partial\lambda}{\partial\varphi} + \frac{2}{r}(\phi_2 \cos\varphi - \phi_1 \sin\varphi)\Big\} \quad (55)$$

$$\sigma_{\theta\varphi} = \mu\Big\{\frac{2\partial^2\lambda}{r^2 \sin\theta\, \partial\theta\, \partial\varphi} - \frac{2\partial}{r\partial\theta}(\phi_2 \cos\varphi - \phi_1 \sin\varphi)$$

$$- \frac{2\partial}{r \sin\theta\, \partial\varphi}[\cos\theta\,(\phi_1 \cos\varphi + \phi_2 \sin\varphi) - \phi_3 \sin\theta]$$

$$- \frac{\cot\theta}{r}\left[\frac{1}{r\sin\theta}\frac{\partial\lambda}{\partial\varphi} - 2(\phi_2 \cos\varphi - \phi_1 \sin\varphi)\right]\Big\} \quad (56)$$

For axisymmetric motion, when $\phi_1 = \phi_2 = 0$ and ϕ_0 and ϕ_3 are independent of φ, equations (49) to (56) simplify considerably. In some simple flow fields e.g. a source, or sink or doublet in a infinite fluid only one of the four harmonic functions, e.g. ϕ_0 is needed to describe the flow. The case of a source in an infinite fluid is considered below.

(c) A source in an infinite fluid: Consider a source of strength m, in an infinite viscous fluid, located at a point C on the z-axis and at a distance h from the origin. For any point P distant r from the origin and such that angle COP $= \theta$,

$CP = \sqrt{r^2 + h^2 - 2rh\cos\theta}$.

The potential of such a flow is

$$\varphi_o = \frac{m}{CP}$$

$$= \frac{m}{\sqrt{r^2 + h^2 - 2rh\cos\theta}}$$

$$= m \sum_{n=0}^{\infty} P_n(\cos\theta)\,\frac{r^n}{h^{n+1}} \tag{57}$$

where $P_n(\cos\theta)$ is the Legendre's function.

For this flow therefore

$$q_r = \frac{\partial\phi_o}{\partial r}$$

$$= m \sum \frac{nr^{n-1}P_n(\cos\theta)}{h^{n+1}} \tag{58}$$

$$q_\theta = \frac{\partial\phi_o}{r\partial\theta} = m \sum \frac{r^{n-1}}{h^{n+1}}\,\frac{d}{d\theta}P_n(\cos\theta)$$

$$= m \sum \frac{r^{n-1}}{h^{n+1}}\,\frac{d\,P_n(\cos\theta)}{d(\cos\theta)}\,\frac{d(\cos\theta)}{d\theta}$$

$$= -m \sum \frac{r^{n-1}}{h^{n+1}}\sin\theta\,P_n' \tag{59}$$

where '/' indicates differentiation with respect to $\cos\theta$.

$$\sigma_{r\theta}^{(o)} = \mu\left\{\frac{\partial q_r}{r\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right\}$$

$$= 2\mu m \sum \frac{(1-n)r^{n-2}\sin\theta\,P_n'}{h^{n+1}} \tag{60}$$

$$p = \frac{2\mu}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\ q_r) + \frac{\partial}{\partial\theta}(r \sin\theta\ q_\theta)\right] = 0 \tag{61}$$

$$\sigma_{rr} = -p + 2\mu \frac{\partial q_r}{\partial r}$$

$$= 2\mu m \sum \frac{n(n-1)r^{n-2}P_n}{h^{n+1}} \tag{62}$$

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r}\left(\frac{\partial q_\theta}{\partial\theta} + q_r\right)$$

$$= 2\mu m \sum \frac{r^{n-2}}{h^{n+1}} \left[n(1-n)P_n + P'_{n-1}\right] \tag{63}$$

$$\sigma_{\varphi\varphi} = -p + \frac{2\mu}{r \sin\theta}\left\{q_r \sin\theta + q_\theta \cos\theta\right\}$$

$$= -2\mu m \sum \frac{r^{n-2}}{h^{n+1}} P'_{n-1} \tag{64}$$

$$\sigma_{r\varphi} = 0 = \sigma_{\theta\varphi} \ .$$

CHAPTER THREE

The flow field of a very thin jet in an infinite viscous fluid

In this chapter we give a further demonstration of how to apply the method given in chapter two to solve a more difficult problem than the one at the end of chapter two. Infact we develop a possible theory for the flow field of a very thin jet passing through an infinite incompressible viscous fluid in steady state. Our only assumption is that the rate of change of momentum of the fluid in the jet is constant.

3.1 Let the plane $z = 0$ be the interface of two immiscible dissimilar fluids both in steady state and extending to infinity on either side of the plane $z = 0$. Let the region $z > 0$ be region 1 and the region $z < 0$ be region 2. Let a circular disc of radius 'a' be placed in region 2 and kept fixed with its circular cross-section at the interface. Consider a jet of momentum Ω impinging directly and uniformly only on the circular cross-section at the interface. Let the change in momentum of the jet with respect to time be constant and equal to F at any instant. By taking the limit as 'a' tends to zero and making the viscosities of the two fluids the same, we should obtain the flow field of a very thin jet passing through an infinite viscous fluid in steady state.

3.2 The appropriate Navier-Stoke's equation is

$$\nabla p = \mu \nabla^2 \underset{\sim}{q}$$

where p pressure, $\underset{\sim}{q}$ = velocity and μ = coefficient of viscosity.

A solution of this equation is, as established earlier in chapter two

$$\underset{\sim}{q} = \nabla(\phi_0 + \underset{\sim}{\rho}\cdot\underset{\sim}{\phi}) - 2\underset{\sim}{\phi}$$
$$p = 2\mu(\nabla\cdot\underset{\sim}{\phi}) \qquad (1)$$

where ϕ_0 and $\underset{\sim}{\phi}$ satisfy the harmonic equations

$$\nabla^2\phi_0 = 0, \quad \nabla^2\underset{\sim}{\phi} = \underset{\sim}{0}$$

and

$$\underset{\sim}{\phi} = (\phi_1, \phi_2, \phi_3).$$

For axisymmetric flows, like the case under consideration, it has been shown that $\phi_1 = \phi_2 = 0$. So that in cylindrical coordinates, equation (1) gives

$$q_z = \frac{\partial}{\partial z}(\phi_0 + z\phi_3) - 2\phi_3 \qquad (2)$$

$$q_\rho = \frac{\partial}{\partial \rho}(\phi_0 + z\phi_3) \qquad (3)$$

and

$$\frac{\partial p}{\partial z} = \mu\,\nabla^2 q_z \qquad (4)$$

The boundary conditions to be satisfied are, at $z = 0$,

$$q_\rho^{(1)} = q_\rho^{(2)} \tag{5}$$

$$q_z^{(1)} = q_z^{(2)} \tag{6}$$

$$\sigma_{\rho z}^{(1)} = \sigma_{\rho z}^{(2)} \tag{7}$$

$$\left.\begin{aligned} \sigma_{zz}^{(2)} - \sigma_{zz}^{(1)} &= \frac{F}{\pi a^2}, \qquad \rho < a \\ \sigma_{zz}^{(2)} &= \sigma_{zz}^{(1)} \qquad\quad \rho > a \end{aligned}\right\} \tag{8}$$

For region (1) i.e. $z > 0$, we take

$$\phi_0^{(1)} = \int_0^\infty A(\lambda)\ J_0(\lambda\rho)e^{-\lambda z}\ d\lambda \tag{9}$$

$$\phi_3^{(1)} = \int_0^\infty B(\lambda)\ J_0(\lambda\rho)e^{-\lambda z}\ d\lambda \tag{10}$$

For region (2) i.e. $z < 0$, we take

$$\phi_0^{(2)} = \int_0^\infty C(\lambda)\ J_0(\lambda\rho)e^{\lambda z}\ d\lambda \tag{11}$$

$$\phi_3^{(2)} = \int_0^\infty D(\lambda)\ J_0(\lambda\rho)e^{\lambda z}\ d\lambda \tag{12}$$

(9) to (12) are solutions of the harmonic equation

$$\frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{13}$$

and they also satisfy the condition that the effect of the disturbance tends to zero at large distances from the obstacle.

From (2), (3), (9) and (10),

$$q_z^{(1)} = \frac{\partial}{\partial z}\left(\phi_0^{(1)} + z\phi_3^{(1)}\right) - 2\phi_3^{(1)}$$

$$= \int_0^\infty -\lambda\, J_0(\lambda\rho)e^{-\lambda z}[A(\lambda) + z\, B(\lambda)]\, d\lambda$$

$$- \int_0^\infty B(\lambda)\, J_0(\lambda\rho)e^{-\lambda z}\, d\lambda \qquad (14)$$

$$q_\rho^{(1)} = \frac{\partial}{\partial \rho}\left(\phi_0^{(1)} + z\phi_3^{(1)}\right)$$

$$= \int_0^\infty -\lambda\, J_1(\lambda\rho)e^{-\lambda z}[A(\lambda) + z\, B(\lambda)]\, d\lambda \qquad (15)$$

From (2), (3), (11) and (12),

$$q_z^{(2)} = \frac{\partial}{\partial z}\left(\phi_0^{(2)} + z\phi_3^{(2)}\right) - 2\phi_3^{(2)}$$

$$= \int_0^\infty \lambda\, J_0(\lambda\rho)e^{\lambda z}[C(\lambda) + z\, D(\lambda)]\, d\lambda$$

$$- \int_0^\infty D(\lambda)\, J_0(\lambda\rho)e^{\lambda z}\, d\lambda \qquad (16)$$

$$q_\rho^{(2)} = \frac{\partial}{\partial \rho}\left(\phi_0^{(2)} + z\, \phi_3^{(2)}\right)$$

$$= \int_0^\infty -\lambda\, J_1(\lambda\rho)e^{\lambda z}[C(\lambda) + z\, D(\lambda)]\, d\lambda \qquad (17)$$

From (14) and (15):

$$\sigma_{\rho z}^{(1)} = \mu_1 \left\{ \frac{\partial q_z^{(1)}}{\partial \rho} + \frac{\partial q_\rho^{(1)}}{\partial z} \right\}$$

$$= 2\mu_1 \int_0^\infty \lambda^2 \; J_1(\lambda\rho) e^{-\lambda z} [A(\lambda) + z\; B(\lambda)] \; d\lambda \tag{18}$$

From (16) and (17):

$$\sigma_{\rho z}^{(2)} = \mu_2 \left\{ \frac{\partial q_z^{(2)}}{\partial \rho} + \frac{\partial q_\rho^{(2)}}{\partial z} \right\}$$

$$= -2\mu_2 \int_0^\infty \lambda^2 J_1(\lambda\rho) e^{\lambda z} [C(\lambda) + z\; D(\lambda)] \; d\lambda \tag{19}$$

Now

$$\frac{\partial q_z^{(1)}}{\partial z} = \int_0^\infty \lambda^2 \; J_0(\lambda\rho) e^{-\lambda z} [A(\lambda) + z\; B(\lambda)] \; d\lambda \tag{20}$$

and

$$\frac{\partial p^{(1)}}{\partial z} = \mu_1 \nabla^2 \; q_z^{(1)}$$

$$= \mu_1 \nabla^2 \left\{ \int_0^\infty - \lambda \; J_0(\lambda\rho) e^{-\lambda z} [A(\lambda) + z\; B(\lambda)] \; d\lambda - \int_0^\infty B(\lambda) \; J_0(\lambda\rho) e^{-\lambda z} \; d\lambda \right\}$$

from (14)

$$= \mu_1 \nabla^2 \left\{ \int_0^\infty - \lambda z \; B(\lambda) \; J_0(\lambda\rho) e^{-\lambda z} \; d\lambda \right\}$$

since $$\nabla^2\left\{ \int_0^\infty - \lambda\ A(\lambda)\ J_o(\lambda\rho)e^{-\lambda z}\ d\lambda - \int_0^\infty B(\lambda)\ J_o(\lambda\rho)e^{-\lambda z}\ d\lambda \right\} = 0$$

$$\therefore \quad \frac{\partial p^{(1)}}{\partial z} = 2\mu_1 \int_0^\infty \lambda^2\ B(\lambda)\ J_o(\lambda\rho)e^{-\lambda z}\ d\lambda \tag{21}$$

$$\Longrightarrow \quad p^{(1)} = 2\mu_1 \int_0^\infty - \lambda\ B(\lambda)\ J_o(\lambda\rho)e^{-\lambda z}\ d\lambda \tag{22}$$

$$\therefore \quad \sigma_{zz}^{(1)} = - p^{(1)} + 2\mu_1 \frac{\partial q_z^{(1)}}{\partial z}$$

$$= 2\mu_1 \int_0^\infty \lambda\ J_o(\lambda\rho)e^{-\lambda z}[\lambda A(\lambda) + B(\lambda)(1+\lambda z)]\ d\lambda \tag{23}$$

Similarly

$$\frac{\partial q_z^{(2)}}{\partial z} = \int_0^\infty \lambda^2 e^{\lambda z}\ J_o(\lambda\rho)[C(\lambda) + z\ D(\lambda)]\ d\lambda \tag{24}$$

and

$$\frac{\partial p^{(2)}}{\partial z} = \mu_2 \nabla^2 q_z^{(2)} = \mu_2 \nabla^2 \left\{ \int_0^\infty \lambda\ J_o(\lambda\rho)e^{\lambda z}[C(\lambda)+z\ D(\lambda)]d\lambda - \int_0^\infty D(\lambda)\ J_o(\lambda\rho)e^{\lambda z}\ d\lambda \right\}$$

from (16)

$$= \mu_2 \nabla^2 \left\{ \int_0^\infty \lambda z\ D(\lambda)e^{\lambda z}\ J_o(\lambda\rho)\ d\lambda \right\}$$

since $$\nabla^2 \left\{ \int_0^\infty \lambda\ C(\lambda)\ J_o(\lambda\rho)e^{\lambda z}d\lambda - \int_0^\infty D(\lambda)\ J_o(\lambda\rho)e^{\lambda z}d\lambda \right\} = 0$$

$$\therefore \quad \frac{\partial p^{(2)}}{\partial z} = 2\mu_2 \int_0^\infty \lambda D(\lambda) e^{\lambda z} J_0(\lambda\rho)\, d\lambda$$

$$\Longrightarrow \quad p^{(2)} = 2\mu_2 \int_0^\infty \lambda D(\lambda) e^{\lambda z} J_0(\lambda\rho)\, d\lambda \tag{25}$$

$$\therefore \quad \sigma_{zz}^{(2)} = -p^{(2)} + 2\mu_2 \frac{\partial q_z^{(2)}}{\partial z}$$

$$= 2\mu_2 \int_0^\infty \lambda J_0(\lambda\rho) e^{\lambda z} [\lambda C(\lambda) + D(\lambda)(\lambda z - 1)] d\lambda \tag{26}$$

From (5), (15) and (17), we have at $z = 0$:

$$\int_0^\infty -\lambda J_1(\lambda\rho)\, A(\lambda) d\lambda = \int_0^\infty -\lambda J_1(\lambda\rho)\, C(\lambda) d\lambda$$

$$\Longrightarrow \quad A(\lambda) = C(\lambda) \tag{27}$$

From (6), (14) and (16) we have at $z = 0$,

$$\int_0^\infty -\lambda\, J_0(\lambda\rho)\, A(\lambda) d\lambda - \int_0^\infty B(\lambda)\, J_0(\lambda\rho) d\lambda$$

$$= \int_0^\infty \lambda J_0(\lambda\rho)\, C(\lambda) d\lambda - \int_0^\infty D(\lambda)\, J_0(\lambda\rho) d\lambda$$

$$\Longrightarrow \quad A(\lambda) = -C(\lambda) \tag{28}$$

and $\qquad B(\lambda) = D(\lambda) \qquad (29)$

From (7), (18) and (19), we have at $z = 0$,

$$2\mu_1 \int_0^\infty \lambda^2 J_1(\lambda\rho) A(\lambda) d\lambda = - 2\mu_2 \int_0^\infty \lambda^2 J_1(\lambda\rho) C(\lambda) d\lambda$$

$$\Longrightarrow \quad \mu_1 A(\lambda) = - \mu_2 C(\lambda) \tag{30}$$

Now $$a \int_0^\infty J_0(\lambda\rho) J_1(\lambda a) d\lambda = \begin{cases} 1, & \text{when } \rho < a \\ 0, & \;,, \quad \rho > a \end{cases}$$

$\therefore$ From (8), (23) and (26), we have at $z = 0$:

$$2 \int_0^\infty \lambda J_0(\lambda\rho) \Big\{ \mu_2 [\lambda C(\lambda) - D(\lambda)] - \mu_1 [\lambda A(\lambda) + B(\lambda)] \Big\} d\lambda$$

$$= \frac{F}{\pi a^2} \cdot a \int_a^\infty J_0(\lambda\rho) J_1(\lambda a)\, d\lambda$$

or $$\int_0^\infty J_0(\lambda\rho) \Big\{ \lambda\mu_2 [\lambda C(\lambda) - D(\lambda)] - \lambda\mu_1 [\lambda A(\lambda) + B(\lambda)]$$

$$- \frac{F}{\pi a} J_1(\lambda a) \Big\} d\lambda = 0$$

$$\Longrightarrow \quad \lambda[\mu_2 \, C(\lambda) - \mu_1 A(\lambda)] - [\mu_2 \, D(\lambda) + \mu_1 B(\lambda)]$$

$$= \frac{F}{\pi a} \cdot \frac{J_1(\lambda a)}{\lambda a} \tag{31}$$

Now from (27) and (28)

$$A(\lambda) = C(\lambda) = - C(\lambda) = 0 \tag{32}$$

Using this in (31) we get

$$\mu_2 \, D(\lambda) + \mu_1 B(\lambda) = - \frac{F}{\pi a} J_1 \frac{(\lambda a)}{\lambda a} \tag{33}$$

In order to get the flow field of a very thin jet passing through an infinite homogeneous fluid, we shall set $\mu_1 = \mu_2 = \mu$ in equation

(33) and then take the limit as $a \to 0$.

Setting $\mu_1 = \mu_2 = \mu$ in (33) we get, on using (29):

$$D = B = -\frac{F}{2\pi\mu} \cdot \frac{J_1(\lambda a)}{\lambda a} \tag{34}$$

Now $$\underset{a \to 0}{\mathrm{Lt}} \frac{J_1(\lambda a)}{\lambda a} = \tfrac{1}{2}$$

$\therefore$ From (34) we have

$$D = B = -\frac{F}{4\pi\mu}$$

so that the potentials for this flow is

$$\phi_0 = 0 \quad \text{and} \quad \varphi_3 = \int_0^\infty -\frac{F}{4\pi\mu} J_0(\lambda\rho)e^{-\lambda z}\, d\lambda \quad \text{from (10)}$$

since $A = C = 0$.

Now $$\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)d\lambda = \frac{1}{\rho}$$

$$\therefore \qquad \phi_3 = -\frac{F}{4\pi\mu\rho} \tag{35}$$

Consequently for a thin jet emanating from a point P along the z-axis at a distance h from the origin, the potential at any point S in the fluid such that $PS = r_1$; angle $POS = \theta$, and $OS = r$, is

$$\phi_3 = \frac{-F}{4\pi\mu} \cdot \frac{1}{r_1}$$

$$= \frac{-F}{4\pi\mu} \cdot \frac{1}{\sqrt{r^2 + h^2 - 2rh\cos\theta}}$$

$$= \frac{-F}{4\pi\mu} \frac{r^n P_n(\cos\theta)}{h^{n+1}} \tag{36}$$

where $P_n(\cos\theta)$ is the Legendre polynomial.

3 Application: Using ϕ_3 we can obtain the stress distribution in the flow field.

Writing $L = \dfrac{-F}{4\pi\mu h^{n+1}}$, ϕ_3 can be written as

$$\phi_3 = L\, r^n\, P_n \tag{37}$$

In spherical coordinates, and from (1),

$$q_r = \frac{\partial}{\partial r}(\phi_0 + \phi_3 r\cos\theta) - 2\phi_3\cos\theta$$

$$= \frac{\partial}{\partial r}(\phi_3 r\cos\theta) - 2\phi_3\cos\theta, \quad \text{since} \quad \phi_0 = 0 .$$

$$\therefore \quad q_r = (n-1)L\, r^n \cos\theta\, P_n$$

$$= \frac{(n-1)L\, r^n\, [(n+1)\, P_{n+1} + nP_{n-1}]}{(2n+1)} \tag{38}$$

$$q_\theta = \frac{\partial}{r\partial\theta}(\phi_0 + \phi_3\, r\cos\theta) + 2\phi_3\sin\theta$$

$$= L\, r^n \sin\theta\, [P_n - \cos\theta\, P_n']$$

where ($'$) indicates differentiation with respect to cos θ

$$q_\theta = L r^n \sin\theta \, [(1-n) P_n - P'_{n-1}]$$

$$= \frac{L r^n \sin\theta}{2n+1} [(n-1) P'_{n+1} + (n+2) P'_{n-1}] \tag{39}$$

$$\sigma_{r\theta} = \mu \left(\frac{\partial q_r}{r\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} \right)$$

$$= \frac{2\mu(n-1)L \sin\theta \, r^{n-1} [nP'_{n+1} + (n+1)P'_{n-1}]}{2n+1} \tag{40}$$

$$p = \frac{2\mu}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta \, q_r) + \frac{\partial}{\partial\theta} (r \sin\theta \, q_\theta) \right] = 0 \tag{41}$$

So that

$$\sigma_{rr} = -p + 2\mu \frac{\partial q_r}{\partial r}$$

$$= \frac{2\mu \, Ln(n-1) r^{n-1} [(n+1) P_{n+1} + nP_{n-1}]}{2n+1} \tag{42}$$

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r} \left(\frac{\partial q_\theta}{\partial\theta} + q_r \right)$$

$$= 2\mu L r^{n-1} \cos\theta \, [n(1-n)P_n + P'_{n-1}] \tag{43}$$

$$\sigma_{\varphi\varphi} = -p + \frac{2\mu}{r \sin\theta} \left(q_r \sin\theta + q_\theta \cos\theta \right)$$

$$= -2\mu L r^{n-1} \cos\theta \, P'_{n-1}$$

$$= 2\mu_1 L r^{n-1} (nP_{n-1} - P'_n) \tag{44}$$

and

$$\sigma_{r\varphi} = 0 = \sigma_{\theta\varphi} \tag{45}$$

CHAPTER FOUR

A Spherical drop of fluid in an infinite dissimilar fluid containing a source.

In this chapter we consider the case when the entire three-dimensional space is filled with two immiscible viscous fluids. One type of fluid occupies the region $r > a$ and a different fluid occupies the region $r < a$ such that their interface $r = a$ is spherical. The region $r > a$ is referred to as the region (1), while the region $r < a$ is referred to as region (2).

A point source of strength m is located in the region (1) at a distance greater than 'a' from the origin. The flow fields in both regions are then investigated with the aim of discovering whether or not any relationship exists between these flow fields and the flow field of a single-fluid space containing a source.

4.1 Our basic assumptions are that a steady-state condition exists, the inertial terms in the Navier-Stoke's equation and body forces are negligible, and that 'a' is sufficiently small and the flow slow enough for the drop of fluid in the region $r < a$ to be nearly spherical.

In the latter part of this chapter, we consider the case when the spherical drop is replaced by a spherical bubble of gas.

4.2 If a steady-state condition exists and if inertial terms and body forces are negligible, the Navier-Stoke's equation for an incompressible viscous fluid is

$$\nabla p = \mu \nabla^2 \underset{\sim}{q} \tag{1}$$

A solution of this equation is as shown in chapter two

$$\begin{aligned} \underset{\sim}{q} &= \nabla(\phi_0 + \underset{\sim}{r}.\underset{\sim}{\phi}) - 2\underset{\sim}{\phi} \\ p &= 2\mu(\nabla.\underset{\sim}{\phi}) \end{aligned} \tag{2}$$

provided $\underset{\sim}{\phi} = (\phi_1, \phi_2, \phi_3)$ and ϕ_0 satisfy the harmonic equations

$$\nabla^2 \underset{\sim}{\phi} = \underset{\sim}{0} \quad \text{and} \quad \nabla^2 \phi_0 = 0$$

respectively.

In spherical coordinates and for axisymmetric flows equations (2) can be written as

$$q_r = \frac{\partial}{\partial r}(\phi_0 + \phi_3 \, r \cos\theta) - 2\phi_3 \cos\theta \tag{3}$$

$$q_\theta = \frac{\partial}{r\partial\theta}(\phi_0 + \phi_3 \, r \cos\theta) + 2\phi_3 \sin\theta \tag{4}$$

$$q_\varphi = 0 \tag{5}$$

$$p = \frac{2\mu}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}(r^2 \sin\theta q_r) + \frac{\partial}{\partial\theta}(r \sin\theta q_\theta)\right) \tag{6}$$

and the stress components can be obtained from the relations

$$\sigma_{rr} = -p + 2\mu \frac{\partial q_r}{\partial r} \tag{7}$$

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r}\left(\frac{\partial q_\theta}{\partial \theta} + q_r\right) \tag{8}$$

$$\sigma_{\varphi\varphi} = -p + \frac{2\mu}{r \sin\theta}\left(q_r \sin\theta + q_\theta \cos\theta\right) \tag{9}$$

$$\sigma_{r\theta} = \mu\left(\frac{\partial q_r}{r\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right) \tag{10}$$

For axisymmetric flows $\sigma_{r\varphi} = \sigma_{\theta\varphi} = 0$.

Now for a source of strength m located at a point $h(> a)$ from the origin along the z-axis, the potential for the flow is, from Chapter 2,

$$\phi^{(0)} = Gr^n P_n(\cos\theta) \tag{11}$$

where $G = m/h^{n+1}$, $P_n(\cos\theta)$ is the Legendre function and superfix (0) refers to flow field in a single-fluid space, n assumes the values from 0 to ∞.

So that for the single-fluid space containing a source of strength m, the velocity and stress components are

$$\begin{aligned} q_r^{(0)} &= n\, Gr^{n-1} P_n \\ q_\theta^{(0)} &= -Gr^{n-1} \sin\theta P_n' \\ q_\varphi^{(0)} &= 0 \end{aligned} \tag{12}$$

$$\sigma_{r\theta}^{(0)} = 2(1-n)\mu^{(1)} Gr^{n-2} \sin\theta\, P_n'$$

where $\mu^{(1)}$ is the viscosity of the fluid in the region $r > a$.

$$\sigma_{rr}^{(o)} = 2n(n-1)\mu^{(1)}Gr^{n-2}P_n$$

$$\sigma_{\theta\theta}^{(o)} = 2\mu^{(1)}r^{n-2}G(n(1-n)P_n + P'_{n-1}) \tag{13}$$

$$\sigma_{\varphi\varphi}^{(o)} = -2\mu^{(1)}r^{n-2}G\,P'_{n-1}\,.$$

4.3 To solve our problem, noting that q_r varies as $P_n(\cos\theta)$, q_θ varies as $P'_n(\cos\theta)$ and also that the components of stress and velocity should vanish as $r \to \infty$, we construct our ϕ_o and ϕ_3 as follows:

For the region $r > a$ (region 1), we have

$$\phi_o^{(1)} = A\,r^{-(n+1)}P_n\,,\quad \phi_o^{(1)} = \frac{B}{a^2}\,r^{-(n+1)}P_n \tag{14a}$$

and for the region $r < a$ (region 2), we have

$$\phi_o^{(2)} = C\,r^nP_n\,,\quad \phi_3^{(2)} = \frac{D}{a^2}\,r^nP_n \tag{14b}$$

where A, B, C, D are to be determined from the boundary conditions

$$\begin{aligned} q_r^{(1)} &= q_r^{(2)} \\ q_\theta^{(1)} &= q_\theta^{(2)} \\ \sigma_{r\theta}^{(1)} &= \sigma_{r\theta}^{(2)} \\ \text{and}\quad \sigma_{rr}^{(1)} &= \sigma_{rr}^{(2)} \end{aligned} \tag{15}$$

at $r = a$ for all θ and φ.

Putting (14a) in equations (3) to (10) and noting that if $q_r^{(*)}$, $q_\theta^{(*)}$, etc are the perturbation velocity components in the region $r > a$, the $q_r^{(1)} = q_r^{(0)} + q_r^{(*)}$, $q_\theta^{(1)} = q_\theta^{(0)} + q_\theta^{(*)}$ etc.,

we have

$$q_r^{(1)} = n\, G r^{n-1} P_n - \frac{(n+1)}{r^{n+2}}\left(A + \frac{n r^2}{a^2} B\right) P_n$$

$$q_\theta^{(1)} = -\, G r^{n-1} \sin\theta\, P_n' - \frac{\sin\theta}{r^{n+3}}\left(A + (n-2)\frac{r^2}{a^2} B\right) P_n'$$

$$\sigma_{r\theta}^{(1)} = 2(1-n)\mu^{(1)} G r^{n-2} \sin\theta\, P_n' + \frac{2\mu^{(1)}\sin\theta}{r^{n+3}}\left((2+n)A + (n^2-1)\frac{r^2}{a^2} B\right) P_n' \tag{16}$$

and

$$\sigma_{rr}^{(1)} = 2\mu^{(1)} n(n-1) G r^{n-2} P_n + \frac{2\mu^{(1)}(n+1)}{r^{n+3}}\left((n+2)\, A + n^2 \frac{r^2}{a^2} B\right) P_n$$

Similarly by putting (14b) in equations (3) to (10) we have for the region $r < a$,

$$q_r^{(2)} = n r^{n-1}\left(C + (n+1)\frac{r^2}{a^2} D\right) P_n$$

$$q_\theta^{(2)} = -\sin\theta\, r^{n-1}\left(C + (n+3)\frac{r^2}{a^2} D\right) P_n'$$

$$\sigma_{r\theta}^{(2)} = 2\mu^{(2)} \sin\theta\, r^{n-2}\left((1-n)C - n(n+2)\frac{r^2}{a^2} D\right) P_n' \tag{17}$$

$$\sigma_{rr}^{(2)} = 2\mu^{(2)}\, n r^{n-2}\left((n-1)C + (n+1)^2\frac{r^2}{a^2} D\right) P_n$$

Applying the boundary conditions (15) to equations (16) and (17) we obtain the following equations:

$$
\begin{aligned}
&nGa^{2n+1} - (n+1)(A+nB) = na^{2n+1}\Big(C + (n+1)D\Big) \\
&Ga^{2n+1} + A+(n-2)B = a^{2n+1}\Big(C + (n+3)D\Big) \\
&(1-n)Ga^{2n+1} + (2+n)A + (n^2-1)B = ga^{2n+1}\Big((1-n)C - n(n+2)D\Big) \\
&n(n-1)Ga^{2n+1} + (n+1)(n+2)A + n^2(n+1)B = gna^{2n+1}\Big((n-1)C + (n+1)^2 D\Big)
\end{aligned}
\tag{18}
$$

where $g = \mu^{(2)}/\mu^{(1)}$.

Solving equations (18) simultaneously

$$
\begin{aligned}
A &= \frac{n(n-1)(2n-1)(1-g)a^{2n+1}G}{(n+1)(1+2g(n-1))} \\
B &= -\frac{(n-1)(2n+1)(1-g)a^{2n+1}G}{(n+1)(1+2g(n-1))} \\
C &= \frac{(2n-1)G}{1+2g(n-1)} \\
D &= 0
\end{aligned}
\tag{19}
$$

That $D = 0$ implies, from (14b), that the flow field in the region $r < a$ is totally described by $\phi_o^{(2)} = Cr^nP_n$.

Now $1 + 2g(n-1) = 2g(n+\alpha)$, where $\alpha = \dfrac{1-2g}{2g}$

$\therefore$ From (16),

$$q_r^{(1)} = nGr^{n-1}P_n - \frac{(n+1)}{r^{n+2}}\left(A + \frac{nr^2}{a^2}B\right)P_n$$

$$= q_r^{(o)} - \frac{nGa^{2n+1}}{r^{n+2}}(\alpha+\tfrac{1}{2})\left[\frac{(n-1)(2n-1)}{n+\alpha} - \frac{r^2}{a^2}\frac{(n-1)(2n+1)}{n+\alpha}\right]P_n$$

$$= q_r^{(\ )} - nGR^{n-1}\left(\frac{a}{r}\right)^3\left(\alpha+\tfrac{1}{2}\right)\Big\{\left(\frac{r^2}{a^2} - 1\right)\left(2\alpha+1\right) - 2$$

$$+ 2n\left(1-\frac{r^2}{a^2}\right) + \frac{(1+\alpha)}{n+\alpha}\left[1+\frac{r^2}{a^2} + 2\alpha\left(1-\frac{r^2}{a^2}\right)\right]\Big\}P_n$$

where

$$R = \frac{a^2}{r}$$

$$\therefore\quad q_r^{(1)}(r,\theta) = q_r^{(o)}(r,\theta) - \left(\frac{a}{r}\right)^3\left(\alpha+\tfrac{1}{2}\right)\Big\{\left[\left(\frac{r^2}{a^2} - 1\right)\left(2\alpha+1\right) - 2\right] q_r^{(o)}(R,\theta)$$

$$- 2\left(1-\frac{r^2}{a^2}\right)\frac{\partial}{\partial R}\left(R\, q_r^{(o)}(R,\theta)\right)$$

$$+ (1+\alpha)\left[1+\frac{r^2}{a^2}+2\alpha\left(1-\frac{r^2}{a^2}\right)\right] \times \tag{20}$$

$$R^{-(\alpha+1)}\int_0^R \lambda^\alpha q_r^{(o)}(\lambda,\theta)\,d\lambda\Big\}$$

Similarly,

$$q_\theta^{(1)} = -Gr^{n-1}\sin\theta\, P_n' - \frac{\sin\theta}{r^{n+2}}\left[A + (n-2)\frac{r^2}{a^2}B\right]P_n'$$

And since A and B could be written as

$$A = \frac{n(n-1)(2n-1)(1-g)a^{2n+1}G}{2g(n+1)(n+\alpha)}$$

$$= a^{2n+1}G(\alpha+\tfrac{1}{2})\left\{-(5+2\alpha)+2n+\frac{\alpha(\alpha+1)(2\alpha+1)}{(\alpha-1)(n+\alpha)}-\frac{6}{(\alpha-1)(n+1)}\right\} \quad (21)$$

and

$$B = -\frac{(n-1)(2n+1)(1-g)a^{2n+1}G}{2g(n+1)(n+\alpha)} = a^{2n+1}G(\alpha+\tfrac{1}{2}) \times$$

$$\left\{-2+\frac{(\alpha+1)(2\alpha-1)}{(\alpha-1)(n+\alpha)}-\frac{2}{(\alpha-1)(n+1)}\right\} \quad (22)$$

$$\therefore \quad q_\theta^{(1)} = q_\theta^{(o)} - \left(\frac{a}{r}\right)^3 R^{n-1}(\alpha+\tfrac{1}{2})\left\{\left[(5+2\alpha)\left(\frac{r^2}{a^2}-1\right)+\frac{2r^2}{a^2}\right]\right.$$

$$+2n\left(1-\frac{r^2}{a^2}\right)+\frac{(\alpha+1)\left[\alpha(2\alpha+1)(1-\frac{r^2}{a^2})+\frac{2r^2}{a^2}(1-\alpha)\right]}{(\alpha-1)(n+\alpha)}$$

$$\left.-\frac{6(1-\frac{r^2}{a^2})}{(\alpha-1)(n+1)}\right\} G\sin\theta\, P_n'$$

$$\therefore \quad q_\theta^{(1)}(r,\theta) = q_\theta^{(o)}(r,\theta) + \left(\frac{a}{r}\right)^3(\alpha+\tfrac{1}{2})\left\{\left[(5+2\alpha)\left(\frac{r^2}{a^2}-1\right)+\frac{2r^2}{a^2}\right] q_\theta^{(o)}(R,\theta)\right.$$

$$+2\left(1-\frac{r^2}{a^2}\right)\frac{\partial}{\partial R}\left(Rq_\theta^{(o)}(R,\theta)\right)+\frac{(\alpha+1)}{\alpha-1}\left[\alpha(2\alpha+1)\left(1-\frac{r^2}{a^2}\right)\right.$$

$$\left.+\frac{2r^2}{a^2}(1-\alpha)R^{-(\alpha+1)}\int_0^R \lambda^\alpha q_\theta^{(o)}(\lambda,\theta)\,d\lambda\right] \quad (23)$$

$$\left.-\frac{6(1-\frac{r^2}{a^2})}{(\alpha-1)}R^{-2}\int_0^R \lambda q_\theta^{(o)}(\lambda,\theta)\,d\lambda\right\}$$

Also

$$\sigma_{r\theta}^{(1)} = 2(1-n)\mu^{(1)}Gr^{n-2}\sin\theta P_n' + \frac{2\mu^{(1)}\sin\theta}{r^{n+3}}[(n+2)A+(n^2-1)\frac{r^2}{a^2}B]$$

and

$$(n+2)A = (\alpha+\tfrac{1}{2})(n-1)a^{2n+1}G\left\{(1-\alpha)+2n+\frac{\alpha(\alpha^2-2\alpha-2)}{(\alpha-1)(n+\alpha)}\right.$$

$$\left.+\frac{3(5-2\alpha)}{(\alpha-1)(n+1)}\right\},$$

$$(n^2-1)B = (\alpha+\tfrac{1}{2})(n-1)a^{2n+1}G\left\{1+2\alpha-2n+\frac{(1+\alpha)(1-2\alpha)}{n+\alpha}\right\}$$

$$\therefore\quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(0)} - 2\mu^{(1)}\left(\frac{a}{r}\right)^5 R^{n-2}G(1-n)(\alpha+\tfrac{1}{2})\left\{[(1-\alpha)+\frac{r^2}{a^2}(1+2\alpha)]+2n(1-\frac{r^2}{a^2})\right.$$

$$\left.+\frac{1}{n+\alpha}[\frac{\alpha(\alpha^2-2\alpha-2)}{\alpha-1}+\frac{r^2}{a^2}(1+\alpha)(1-2\alpha)]+\frac{3(5-2\alpha)}{(\alpha-1)(n+1)}\right\}\sin\theta\, P_n'$$

$$\therefore\quad \sigma_{r\theta}^{(1)}(r,\theta) = \sigma_{r\theta}^{(0)}(r,\theta) - \left(\frac{a}{r}\right)^5(\alpha+\tfrac{1}{2})\left\{[(1-\alpha)+\frac{r^2}{a^2}(1+2\alpha)]\sigma_{r\theta}^{(0)}(R,\theta)\right.$$

$$+\,2(1-\frac{r^2}{a^2})R^{-1}\frac{\partial}{\partial R}(R^2\sigma_{r\theta}^{(0)}(R,\theta))$$

$$+\,[\frac{\alpha(\alpha^2-2\alpha-2)}{\alpha-1}+\frac{r^2}{a^2}(1+\alpha)(1-2\alpha)]R^{-(\alpha+2)}\int_0^R \lambda^{\alpha+1}\sigma_{r\theta}^{(0)}(\lambda,\theta)d\lambda \tag{24}$$

$$\left.+\,\frac{3(5-2\alpha)}{\alpha-1}R^{-3}\int_0^R \lambda^2\sigma_{r\theta}^{(0)}(\lambda,\theta)d\lambda\right\}$$

Similarly

$$\sigma_{rr}^{(1)} = 2\mu^{(1)} n(n-1) G r^{n-2} P_n + \frac{2\mu^{(1)}(n+1)}{r^{n+3}} [(n+2)A + n^2 \frac{r^2}{a^2} B] P_n$$

$$= \sigma_{rr}^{(0)} + 2\mu^{(1)} n(n-1) R^{n-2} (\frac{a}{r})^5 G(\alpha+\tfrac{1}{2}) \Big\{ 2 + (1-\frac{r^2}{a^2})(1-2\alpha) + 2n(1-\frac{r^2}{a^2})$$

$$- \frac{1}{n+\alpha}[2(1+\alpha) + \alpha(1-2\alpha)(1-\frac{r^2}{a^2})] \Big\} P_n$$

$$\therefore \quad \sigma_{rr}^{(1)}(r,\theta) = \sigma_{rr}^{(0)}(r,\theta) + (\frac{a}{r})^5 (\alpha+\tfrac{1}{2}) \Big\{ [2+(1-\frac{r^2}{a^2})(1-2\alpha)] \sigma_{rr}^{(0)}(R,\theta)$$

$$+ 2(1-\frac{r^2}{a^2}) R^{-1} \frac{\partial}{\partial R}(R^2 \sigma_{rr}^{(0)}(R,\theta)) \qquad (25)$$

$$- [2(1+\alpha)+\alpha(1-2\alpha)(1-\frac{r^2}{a^2})] R^{-(\alpha+2)} \int_0^R \lambda^{\alpha+1} \sigma_{rr}^{(0)}(\lambda,\theta) d\lambda \Big\}$$

$$\sigma_{\theta\theta}^{(1)} = 2\mu^{(1)} G r^{n-2} [n(1-n) P_n + P'_{n-1}] + 2\mu^{(1)} r^{-(n+3)} \Big\{ A[P'_{n-1} - (n^2+n+1) P_n]$$

$$- \frac{r^2}{a^2} B[(2-n) P'_{n-1} + 3n(n^2-3n+1) P_n] \Big\}$$

$$= \sigma_{\theta\theta}^{(0)} + 2\mu^{(1)} G R^{n-2} (\frac{a}{r})^5 (\alpha+\tfrac{1}{2}) \Big\{ [-(1+2\alpha) + 2n + \frac{2\alpha^3 - \alpha^2 + \alpha}{(\alpha-1)(n+\alpha)} - \frac{3}{(\alpha-1)(n+1)}$$

$$+ \frac{r^2}{a^2}\left(4 - 2n + \frac{2-2\alpha^3-5\alpha^2-\alpha}{(\alpha-1)(n+\alpha)} + \frac{6}{(\alpha-1)(n+1)}\right)][n(1-n)P_n + P'_{n-1}] + [\frac{1}{(\alpha-1)(n+\alpha)}$$

$$+ \frac{r^2}{a^2}(17 - 6\alpha - 4n + \frac{3(7-4\alpha)}{(\alpha-1)(n+1)}) + \frac{1-4\alpha^3-10\alpha^2+4\alpha}{(\alpha-1)(n+\alpha)}] \, n(1-n) P_n$$

$$+ [-4 + \frac{4\alpha^2}{(\alpha-1)(n+\alpha)} - \frac{3}{(\alpha-1)(n+1)} + (2\alpha+3)\frac{r^2}{a^2}] P'_{n-1} \Big\}$$

$$\therefore \quad \sigma^{(1)}_{\theta\theta}(r,\theta) = \sigma^{(0)}_{\theta\theta}(r,\theta)+\left(\frac{a}{r}\right)^5(\alpha+\tfrac{1}{2})\Big\{\left(4\frac{r^2}{a^2}-1-2\alpha\right)\sigma^{(0)}_{\theta\theta}(R,\theta)$$

$$+\ 2\left(1-\frac{r^2}{a^2}\right)R^{-1}\frac{\partial}{\partial R}\left(R^2\sigma^{(0)}_{\theta\theta}(R,\theta)\right)+\left[\frac{\left(1-\frac{r^2}{a^2}\right)(2\alpha^3+\alpha)-\alpha^2+\frac{r^2}{a^2}(2-5\alpha^2)}{\alpha-1}\right]\times$$

$$R^{-(\alpha+2)}\int_0^R\lambda^{\alpha+1}\sigma^{(0)}_{\theta\theta}(\lambda,\theta)d\lambda-\frac{3}{(\alpha-1)}\left(1-\frac{2r^2}{a^2}\right)R^{-3}\int_0^R\lambda^2\sigma^{(0)}_{\theta\theta}(\lambda,\theta)d\lambda$$

$$-\ \Big[(17-6\alpha)\frac{r^2}{a^2}\sigma^{(0)}_{rr}(R,\theta)-\frac{4r^2}{a^2}R^{-1}\frac{\partial}{\partial R}\left(R^2\sigma^{(0)}_{rr}(R,\theta)\right)$$

$$+\ \frac{1}{\alpha-1}\left(1+\frac{r^2}{a^2}(1-4\alpha^3-10\alpha^2+4\alpha)\right)R^{-(\alpha+2)}\int_0^R\lambda^{\alpha+1}\sigma^{(0)}_{rr}(\lambda,\theta)d\lambda \tag{26}$$

$$+\ \frac{3r^2(7-4\alpha)}{a^2(\alpha-1)}R^{-3}\int_0^R\lambda^2\sigma^{(0)}_{rr}(\lambda,\theta)d\lambda\Big]$$

$$-\ \Big[\left((2\alpha+3)\frac{r^2}{a^2}-4\right)\sigma^{(0)}_{\varphi\varphi}(R,\theta)+\frac{4\alpha^2}{\alpha-1}R^{-(\alpha+2)}\int_0^R\lambda^{\alpha+1}\sigma^{(0)}_{\varphi\varphi}(\lambda,\theta)d\lambda$$

$$-\ \frac{3}{\alpha-1}R^{-3}\int_0^R\lambda^2\sigma^{(0)}_{\varphi\varphi}(\lambda,\theta)d\lambda\Big]\Big\}$$

$$\sigma^{(1)}_{\varphi\varphi} = -2\mu^{(1)}r^{n-1}GP'_{n-1}-2\mu^{(1)}r^{-(n-3)}\Big\{\Big[(2n+1)A$$

$$+\ n(2n-1)\frac{r^2}{a^2}B\Big]P_n+\left[A+(n-2)\frac{r^2}{a^2}B\right]P'_{n-1}\Big\}$$

$$= \sigma_{\varphi\varphi}^{(o)} + (\alpha+\tfrac{1}{2})(\tfrac{a}{r})^{5} R^{n-2} G\Big\{(1-\tfrac{r^2}{a^2})[4 + \frac{3}{(\alpha-1)(n+1)}$$

$$- \frac{(1+\alpha)(2\alpha-1)}{(\alpha-1)(n+\alpha)}]\, n(n-1)P_n + [-(5+2\alpha)$$

$$+ \frac{r^2}{a^2}(7+2\alpha) + 2n(1-\frac{r^2}{a^2}) + 6\,\frac{(\frac{r^2}{a^2}-1)}{(\alpha-1)(n+1)}$$

$$+ \frac{(\alpha+1)[\alpha(2\alpha+1) - \frac{r^2}{a^2}(2\alpha-1)(2+\alpha)]}{(\alpha-1)\;(n+\alpha)}\;]\; P'_{n-1}$$

$$\therefore \quad \sigma_{\varphi\varphi}^{(1)}(r,\theta) = \sigma_{\varphi\varphi}^{(o)}(r,\theta) + (\alpha+\tfrac{1}{2})(\tfrac{a}{r})^{5}\Big\{[2+(\tfrac{r^2}{a^2}-1)(2\alpha+5)]\sigma_{\varphi\varphi}^{(o)}(R,\theta)$$

$$+ 2(1-\frac{r^2}{a^2})R^{-1}\frac{\partial}{\partial R}(R^2\sigma_{\varphi\varphi}^{(o)}(R,\theta))$$

$$+ \frac{(\alpha+1)}{\alpha-1}[\alpha(2\alpha+1)-\frac{r^2}{a^2}(2\alpha-1)(2+\alpha)]R^{-(\alpha+2)}\int_0^R \lambda^{\alpha+1}\sigma_{\varphi\varphi}^{(o)}(\lambda,\theta)d\lambda \qquad (27)$$

$$+ \frac{6}{(\alpha-1)}(\frac{r^2}{a^2}-1)R^{-3}\int_0^R \lambda^2\sigma_{\varphi\varphi}^{(o)}(\lambda,\theta)d\lambda + (1-\frac{r^2}{a^2})[4\sigma^{(o)}(R,\theta)$$

$$+ \frac{3R^{-3}}{\alpha-1}\int_0^R \lambda^2\sigma_{rr}^{(o)}(\lambda,\theta)d\lambda - \frac{(1+\alpha)(2\alpha-1)}{\alpha-1}\,R^{-(\alpha+2)}\int_0^R \lambda^{\alpha+1}\sigma_{rr}^{(o)}(\lambda,\theta)d\lambda]\Big\}$$

Similarly,

$$q_r^{(2)} = nC\, r^{n-1}P_n = nG\, r^{n-1}g^{-1}(1-\frac{\alpha+\frac{1}{2}}{n+\alpha})P_n$$

$$\therefore \quad q_r^{(2)}(r,\theta) = g^{-1}[q_r^{(o)}(r,\theta) - (\alpha+\tfrac{1}{2})r^{-(\alpha+1)}\int_0^r \lambda^{\alpha}q_r^{(o)}(\lambda,\theta)d\lambda] \qquad (28)$$

$$q_\theta^{(2)} = -\,C \sin\theta\, r^{n-1}$$

$$= -\,r^{n-1} G g^{-1} \left(1 - \frac{\alpha + \frac{1}{2}}{n + \alpha}\right) \sin\theta\, P_n'$$

$$\therefore\quad q_\theta^{(2)}(r,\theta) = g^{-1}\left[q_\theta^{(0)}(r,\theta) - (\alpha+\tfrac{1}{2}) r^{-(\alpha+1)} \int_0^r \lambda^\alpha q_\theta^{(0)}(\lambda,\theta)\,d\lambda\right] \tag{29}$$

$$\sigma_{r\theta}^{(2)} = 2\mu^{(2)} r^{n-2}(1-n)\, C \sin\theta\, P_n'$$

$$= 2\mu^{(1)}(1-n) G r^{n-2}\left(1 - \frac{\alpha+\frac{1}{2}}{n+\alpha}\right) \sin\theta\, P_n'$$

$$\therefore\quad \sigma_{r\theta}^{(2)}(r,\theta) = \sigma_{r\theta}^{(0)}(r,\theta) - (\alpha+\tfrac{1}{2}) r^{-(\alpha+2)} \int_0^r \lambda^{\alpha+1} \sigma_{r\theta}^{(0)}(\lambda,\theta)\,d\lambda \tag{30}$$

$$\sigma_{rr}^{(2)} = 2\mu^{(2)} n(n-1) r^{n-2} C\, P_n = 2\mu^{(1)} n(n-1) r^{n-2} G\left(1 - \frac{\alpha+\frac{1}{2}}{n+\alpha}\right) P_n$$

$$\therefore\quad \sigma_{rr}^{(2)}(r,\theta) = \sigma_{rr}^{(0)}(r,\theta) - (\alpha+\tfrac{1}{2}) r^{-(\alpha+2)} \int_0^r \lambda^{\alpha+1} \sigma_{rr}^{(0)}(\lambda,\theta)\,d\lambda \tag{31}$$

$$\sigma_{\theta\theta}^{(2)} = 2\mu^{(2)} r^{n-2} C\left[n(1-n)P_n + P_{n-1}'\right]$$

$$= 2\mu^{(1)} r^{n-2} G\left(1 - \frac{\alpha+\frac{1}{2}}{n+\alpha}\right)\left[n(1-n)P_n + P_{n-1}'\right]$$

$$\therefore\quad \sigma_{\theta\theta}^{(2)}(r,\theta) = \sigma_{\theta\theta}^{(0)}(r,\theta) - (\alpha+\tfrac{1}{2}) r^{-(\alpha+2)} \int_0^r \lambda^{\alpha+1} \sigma_{\theta\theta}^{(0)}(\lambda,\theta)\,d\lambda \tag{32}$$

and

$$\sigma_{\varphi\varphi}^{(2)} = -\,2\mu^{(2)} r^{n-2} C\, P'_{n-1} = -\,2\mu^{(1)} r^{n-2} G \left(1 - \frac{\alpha+\frac{1}{2}}{n+\alpha}\right) P'_{n-1} \tag{33}$$

$$\therefore\quad \sigma_{\varphi\varphi}^{(2)}(r,\theta) = \sigma_{\varphi\varphi}^{(0)}(r,\theta) - (\alpha+\tfrac{1}{2})\, r^{-(\alpha+2)} \int_0^r \lambda^{\alpha+1} \sigma_{\varphi\varphi}^{(0)}(\lambda,\theta)\, d\lambda$$

4.3 In the special case of a gas bubble, the boundary conditions are

$$q_r^{(1)} = q_r^{(2)} = 0,$$

$$q_\theta^{(1)} = q_\theta^{(2)} \quad \text{and}$$

$$\sigma_{r\theta}^{(1)} \quad \sigma_{r\theta}^{(2)} .$$

Applying these boundary conditions we have the following equations:

$$A + n\,B = \frac{nG}{n+1}\, a^{2n+1}$$

$$C' = -(n+1)D \tag{34}$$

$$A + (n-2)\,B = (2D - G)a^{2n+1}$$

$$(2+n)A + (n^2-1)\,B = -\,a^{2n+1}[(1-n)G + g(1+2n)D.$$

The solution of (34) is

$$A = \frac{n(1 - 2n)g\, Ga^{2n+1}}{2(n + 1)(1 + g)} \tag{35}$$

$$B = \frac{G(2 + g + 2ng)a^{2n+1}}{2(n + 1)(1 + g)} \tag{36}$$

$$C = \frac{(1 - 2n)G}{2(1 + g)} \tag{37}$$

$$D = \frac{(2n - 1)G}{2(n+1)(1+g)} \tag{38}$$

Using the relations

$$\sigma_{rr} = -p + 2\mu \frac{\partial q_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r}\left(\frac{\partial q_\theta}{\partial \theta} + q_r\right)$$

$$\sigma_{\varphi\varphi} = -p + \frac{2\mu}{r \sin\theta}\left(q_r \sin\theta + q_\theta \cos\theta\right)$$

where

$$p = \frac{2\mu}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, q_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, q_\theta)\right]$$

we can obtain the other stress components. For the homogenous fluid these are

$$\sigma_{rr}^{(o)} = 2\mu_1\, n(n-1)\, G\, r^{n-2} P_n \tag{39}$$

$$\sigma_{\theta\theta}^{(o)} = 2\mu_1\, r^{n-2}\, G[n(1-n)P_n + P'_{n-1}] \tag{40}$$

$$\sigma_{\varphi\varphi}^{(o)} = -2\mu_1 r^{n-2} G\, P'_{n-1} \tag{41}$$

$$\sigma_{r\varphi} = \sigma_{\theta\varphi} = 0.$$

We shall take note of the following relations:

Let

$$R = a^2/r \tag{42}$$

and let

$$\bar{q}_r^o = q_r^o\,(\theta, R)$$

i.e. $$\bar{q}_r^o = n\, GR^{n-1} P_n, \qquad = n\, G \left(\frac{a^2}{r}\right)^{n-1} P_n \tag{43}$$

from (9)

Then

$$R\, \bar{q}_r^o = n\, GR^n P_n = n\, G\left(\frac{a^2}{r}\right)^n P_n$$

$$\therefore \frac{\partial}{\partial r}\left(R\, \bar{q}_r^o\right) = -\, n^2 G\left(\frac{a^2}{r}\right)^{n-1} \frac{a^2}{r^2} P_n \tag{44}$$

Also

$$\int_0^{a^2/r} \left(R\, \bar{q}_r^o\right) dR = \frac{n\, G}{n+1} \left(\frac{a^2}{r}\right)^{n-1} \frac{a^4}{r^2} P_n \tag{45}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\, \bar{q}_r^o\right) = -\, n^2 G\, a^{2(n-1)} P_n / r^{n+1} \tag{46}$$

and

$$\int_0^r \left(\lambda\, q_r^o(\theta,\lambda)\right) d\lambda = \frac{n\, G}{n+1}\, r^{n-1} . r^2 P_n \tag{47}$$

$$\frac{\partial}{\partial r}\left(r\, q_r^o\right) = n^2 G\, r^{n-1} P_n \tag{48}$$

Similarly, let

$$\bar{q}_\theta^o = q_\theta^o\, (\theta, R)$$

$$= -\, G\, R^{n-1} \sin\theta\, P_n', \quad \text{from (10)}$$

$$= -\, G \left(\frac{a^2}{r}\right)^{n-1} \sin\theta\, P_n' \tag{49}$$

then

$$\frac{\partial}{\partial r}\left(R\, \bar{q}_\theta^o\right) = n\, G\left(\frac{a^2}{r}\right)^{n-1} \cdot \frac{a^2}{r^2} \sin\theta\, P_n' \tag{50}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\sigma^o_{r\theta}\right) = 2n(1-n)\mu_1 \; G \; r^{n-2} \sin\theta \; P_n' \tag{59}$$

$$\frac{1}{r}\int_o^r \left(\lambda^2 \; \sigma^o_{r\theta}(\theta,\lambda)\right) d\lambda = \frac{2(1-n)\mu_1 \; G \; r^n \sin\theta \; P_n'}{n+1} \tag{60}$$

Then

$$q_r^{(1)} = q_r^{(o)}(\theta,r) - \frac{a[a^2 g + r^2(2+g)] \; q_r^o(\theta,R)}{2(1+g)r^3}$$

$$+ \frac{ag(r^2-a^2)}{r(1+g)} \; \frac{\partial}{\partial r}\left[\frac{1}{r} \; q_r^o(\theta,R)\right] \tag{61}$$

from (33) and (34)

Similarly

$$q_\theta^{(1)} = q_\theta^{(o)}(\theta,r) + \frac{a[3a^2 g + r^2(2-5g)]}{2(1+g)r^3} \; q_\theta^{(o)}(\theta,R)$$

$$- \frac{ag(r^2-a^2)}{r(1+g)} \; \frac{\partial}{\partial r}\left[\frac{1}{r} \; q_\theta^{(o)}(\theta,R)\right]$$

$$+ \frac{3[g(r^2-a^2) - 2r^2]}{2(1+g)r \; a^3} \int_o^{a^2/r} [R \; q_\theta^{(o)}(\theta,R)] dR \tag{62}$$

$$\sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(0)}(\theta,r) + \frac{a^3[2a^2 g - r^2(2+g)]}{2(1+g)r^5}\,\sigma_{r\theta}^{(0)}(\theta,R)$$

$$- \frac{g(a^2-r^2)a^3}{(1+g)r^2}\,\frac{\partial}{\partial r}\left[\frac{1}{r^3}\,\sigma_{r\theta}^{(0)}(\theta,R)\right]$$

$$- \frac{g}{(1+g)ar^2}\int_0^{a^2/r}[R^2\sigma_{r\theta}^{(0)}(\theta,R)]dR$$

$$+ \frac{\mu_1 a^3 g}{r^4(1+g)}\left\{q_\theta^{(0)}(\theta,R) - r^2\frac{\partial}{\partial r}\left[\frac{1}{r}\,q_\theta^{(0)}(\theta,R)\right]\right.$$

$$\left. - \frac{r^2}{a^4}\int_0^{a^2/r}[R\;q_\theta^{(0)}(\theta,R)]dR\right\} \tag{63}$$

and

$$q_r^{(2)} = \frac{1}{2(1+g)}\left(1-\frac{r^2}{a^2}\right)q_r^{(0)}(\theta,r)$$

$$+ \frac{1}{1+g}\left(\frac{r^2}{a^2}-1\right)\frac{\partial}{\partial r}[r\;q_r^{0}(\theta,r)] \tag{64}$$

$$q_\theta^{(2)} = \frac{1}{2(1+g)}\left(1+\frac{3r^2}{a^2}\right)q_\theta^{(0)}(\theta,r) + \frac{1}{1+g}\left(\frac{r^2}{a^2}-1\right)\frac{\partial}{\partial r}[r\;q_\theta^{(0)}(\theta,r)]$$

$$- \frac{3}{(1+g)a^2}\int_0^r[\lambda\;q_\theta^{(0)}(\theta,\lambda)]d\lambda \tag{65}$$

$$\sigma_{r\theta}^{(2)} = \frac{g}{2(1+g)}\left(1+\frac{2r^2}{a^2}\right)\sigma_{r\theta}^{(0)}(\theta,r)$$

$$+ \frac{g}{(1+g)r}\left(\frac{r^2}{a^2}-1\right)\frac{\partial}{\partial r}[r^2\sigma_{r\theta}^{(0)}(\theta,r)]$$

$$- \frac{g}{a^2 r(1+g)} \int_0^r [\lambda^2 \sigma_{r\theta}^{(0)}(\theta,\lambda)] d\lambda$$

$$+ \frac{\mu_1 g r}{(1+g)a^2} \left\{ q_\theta^{(0)}(\theta,r) + \frac{\partial}{\partial r} [r\ q_\theta^{(0)}(\theta,r)] \right.$$

$$\left. - \frac{1}{r^2} \int_0^r [\lambda\ q_\theta^{(0)}(\theta,\lambda)] d\lambda \right\} \qquad (66)$$

The following relationships also exist between the other stress components of the undisturbed field and those outside and inside the sphere.

$$\sigma_{rr}^{(1)} = \sigma_{rr}^{(0)}(\theta,r) + \frac{a^3[(2+g)r^2 - 4a^2 g]}{2(1+g)r^5} \sigma_{rr}^{(0)}(\theta,R)$$

$$- \frac{a^3 g(r^2 - a^2)}{(1+g)r^3} \frac{\partial}{\partial r} [\frac{1}{r^2} \sigma_{rr}^{(0)}(\theta,R)]$$

$$+ \frac{\mu_1 a[(2+g)r^2 - 2a^2 g]}{(1+g)r^4} q_r^{(0)}(\theta,R)$$

$$- \frac{\mu_1 a g(2r^2 - a^2)}{(1+g)r^2} \frac{\partial}{\partial r} [\frac{1}{r} q_r^{(0)}(\theta,R)] \qquad (67)$$

$$\sigma_{\theta\theta}^{(1)} = \sigma_{\theta\theta}^{(0)}(\theta,r) + \frac{a^3}{2(1+g)r^5}\Big\{[2r^2 + g(3a^2-r^2)]\ \sigma_{\theta\theta}^{(0)}(\theta,R)$$

$$- \ 2g(r^2-a^2)r^3\ \frac{\partial}{\partial r}[\frac{1}{r^2}\ \sigma_{\theta\theta}^{(0)}(\theta,R)]$$

$$- [2r^2 + g(3a^2 - r^2)]\frac{r^3}{a^6}\int_0^{a^2/r} [R^2\sigma_{\theta\theta}^{(0)}(\theta,R)]dR$$

$$+ \ 2[2r^2 - g(7r^2 + 2a^2)]\ \sigma_{rr}^{(0)}(\theta,R)$$

$$- 4gr^5\ \frac{\partial}{\partial r}[\frac{1}{r^2}\ \sigma_{rr}^{(0)}(\theta,R)]$$

$$+ \ 2[g(4r^2 + a^2) - 8r^2]\ \frac{r^3}{a^6}\int_0^{a^2/r} [R^2\sigma_{rr}^{(0)}(\theta,R)]dR\Big\}$$

$$- \frac{a^3}{(1+g)r^3}\Big\{2g\sigma_{\varphi\varphi}^{(0)}(\theta,R) + (2-g)\ \frac{r^3}{a^6}\int_0^{a^2/r} [R^2\sigma_{\varphi\varphi}^{(0)}(\theta,R)]dR\Big\}$$

$$+ \frac{\mu_1 a}{(1+g)r^4}\Big\{2g(a^2 + 3r^2)\ q_r^{(0)}(\theta,R)$$

$$+ [6r^2 - g(a^2 + 3r^2)]\ \frac{r^2}{a^4}\int_0^{a^2/r} [R\ q_r^{(0)}(\theta,R)]dR\Big\} \qquad (68)$$

$$\sigma^{(1)}_{\varphi\varphi} = \sigma^{(0)}_{\varphi\varphi}(\theta,r) - \frac{a^3}{2(1+g)r^5}\Big\{(5gr^2 - 3a^2 g - 2r^2)\sigma^{(0)}_{\varphi\varphi}(\theta,R)$$

$$- 2g(a^2-r^2)r^3 \frac{\partial}{\partial r}[\frac{1}{r^3}\, \sigma^{(0)}_{\varphi\varphi}(\theta,R)]$$

$$+ [3r^2(g-2) - 3a^2 g]\frac{r^3}{a^6}\int_0^{a^2/r}[R^2\sigma^{(0)}_{\varphi\varphi}(\theta,R)]dR\Big\}$$

$$- \frac{ga^3}{(1+g)r^5}\Big\{2(a^2 - gr^2)\sigma^{(0)}_{rr}(\theta,R)$$

$$- [a^2 + r^2(2-g)]\frac{r^3}{a^6}\int_0^{a^2/r}[R^2\sigma^{(0)}_{rr}(\theta,R)]dR\Big\} \tag{69}$$

$$- \frac{\mu_1 a}{(1+g)r^4}\Big\{2g(r^2-a^2)\, q^{(0)}_r(\theta,R)$$

$$+ [a^2 g + r^2(2-g)]\frac{r^2}{a^4}\int_0^{a^2/r}[Rq^{(0)}_r(\theta,R)]dR\Big\}$$

Similarly

$$\sigma^{(2)}_{rr} = \frac{g}{2(1+g)}(1+\frac{2r^2}{a^2})\, \sigma^{(0)}_{rr}(\theta,r)$$

$$+ \frac{g}{(1+g)r}(\frac{r^2}{a^2} - 1)\frac{\partial}{\partial r}[r^2\sigma^{(0)}_{rr}(\theta,r)] \tag{70}$$

$$+ \frac{\mu_1 gr}{(1+g)a^2}\Big\{q^{(0)}_r(\theta,r) + \frac{\partial}{\partial r}[rq^{(0)}_r(\theta,r)]\Big\}$$

$$\sigma_{\theta\theta}^{(2)} = \frac{g}{1+g}\left(1-\frac{r^2}{a^2}\right)\left\{\frac{\sigma_{\theta\theta}^{(0)}(\theta,r)}{2} - \frac{1}{r}\frac{\partial}{\partial r}\left[r^2\sigma_{\theta\theta}^{(0)}(\theta,r)\right]\right\}$$

$$- \frac{2g}{1+g}\cdot\frac{r^2}{a^2}\left\{\sigma_{rr}^{(0)}(\theta,r) - \frac{1}{r^3}\int_0^{\lambda}\left[\lambda^2\sigma_{rr}^{(0)}(\theta,\lambda)\right]d\lambda\right\}$$

$$- \frac{gr^2}{(1+g)a^2}\left\{2\sigma_{\varphi\varphi}^{(0)}(\theta,r) - \frac{3}{r^3}\int_0^{r}\left[\lambda^2\sigma_{\varphi\varphi}^{(0)}(\theta,\lambda)\right]d\lambda\right\} \tag{71}$$

$$- \frac{2\mu_1 gr}{(1+g)a^2}\left\{q_r^{(0)}(\theta,r) - \frac{1}{r^2}\int_0^{r}\left[\lambda q_r^{(0)}(\theta,\lambda)\right]d\lambda\right\}$$

$$\sigma_{\varphi\varphi}^{(2)} = \frac{g}{1+g}\left(1-\frac{r^2}{a^2}\right)\left\{\frac{\sigma_{\varphi\varphi}^{(0)}(\theta,r)}{2} - \frac{1}{r}\frac{\partial}{\partial r}\left[r^2\sigma_{\varphi\varphi}^{(0)}(\theta,r)\right]\right\}$$

$$+ \frac{gr^2}{(1+g)a^2}\left\{2\sigma_{\varphi\varphi}^{(0)}(\theta,r) - \frac{3}{r^3}\int_0^{r}\left[\lambda^2\sigma_{\varphi\varphi}^{(0)}(\theta,\lambda)\right]d\lambda\right\} \tag{72}$$

$$- \frac{2g}{(1+g)ra^2}\int_0^{r}\left[\lambda^2\sigma_{rr}^{(0)}(\theta,\lambda)\right]d\lambda - \frac{2\mu_1 g}{(1+g)ra^2}\int_0^{r}\left[\lambda q_r^{(0)}(\theta,\lambda)\right]d\lambda$$

CHAPTER FIVE

Slow two-phase shear-flow with a spherical interface

Our second comparison problem is considered in this chapter. It is a non-axisymmetric flow problem and our aim here is to investigate if we could draw conclusions similar to those we drew in an axisymmetric flow problem.

Our assumptions concerning the size of the spherical drop and the boundary conditions will be the same as in chapter four.

5.1 In the first part of this chapter we consider the problem of two immiscible incompressible viscous fluids occupying the regions $r > a$ and $r < a$ respectively of the three-dimensional space. Here r, θ, φ denote the spherical polar co-ordinates. The fluid in the region $r > a$, otherwise called region (1), is undergoing shear flow.

We investigate if any relationships exist between results for the two-phase flow. In the second part, we consider the case when the fluid in the region $r < a$, otherwise called region (2), is replaced by a spherical gas bubble of negligible viscosity. Our aim is to find out if the conclusion of the first part remains valid. In both cases the force tending to deform the sphere can be obtained from the results of the single-phase flow.

5.2 Assume first of all that the entire three-dimensional space is filled with an incompressible viscous fluid undergoing shear flow which could be described by

$$u = \tfrac{1}{2}\alpha x , \quad v = -\tfrac{1}{2}\alpha y , \quad w = 0 \tag{1}$$

in Cartesian coordinates. In spherical coordinates (1) is equivalent to

$$q_r^{(o)} = \tfrac{1}{2}\alpha r \sin^2\theta \cos 2\varphi = kr \cos 2\varphi \, P_2^2 \tag{2}$$

where $k = \alpha/6$, superfix (O) refers to original flow field and

$$P_2^2 = 3 \sin^2\theta$$

$$q_\theta^{(o)} = \tfrac{1}{2}\alpha r \sin\theta \cos\theta \cos 2\varphi = \frac{kr \cos 2\varphi}{2} \frac{\partial P_2^2}{\partial\Theta} \tag{3}$$

and $$q_\varphi^{(o)} = -\tfrac{1}{2}\alpha r \sin\theta \sin 2\varphi = -\frac{kr \sin 2\varphi \, P_2^2}{\sin\theta} \tag{4}$$

so that the generating potential for the flow is

$$\phi^{(o)} = \frac{1}{4}\alpha r^2 \sin^2\theta \cos 2\varphi = \frac{k}{2} r^2 \cos 2\varphi \, P_2^2 \tag{5}$$

since $q_r^{(o)} = \dfrac{\partial\phi^{(o)}}{\partial r}$, $q_\theta^{(o)} = \dfrac{\partial\phi^{(o)}}{r\partial\theta}$ and $q_\varphi^{(o)} = \dfrac{1}{r\sin\theta}\dfrac{\partial\phi^{(o)}}{\partial\varphi}$.

Now let a small spherical drop of a dissimilar fluid be introduced into the original fluid with its centre at the origin.

On the introduction of the spherical drop, the field external to the drop shall be designated by superfix (1) and the field internal to the drop by superfix (2).

The superfix (*) shall refer to the perturbation of the external fluid by the drop.

5.3 The appropriate Navier-Stoke's equation for the problem is, on neglicting inertial terms and body forces,

$$\nabla p = \mu \nabla^2 \underset{\sim}{q} \tag{6}$$

the solution to which is (as established in an earlier paper),

$$\underset{\sim}{q} = \nabla (\phi_0 + \underset{\sim}{r}\cdot\underset{\sim}{\phi}) - 2\underset{\sim}{\phi} , \quad p = 2\mu(\nabla\cdot\underset{\sim}{\phi}) \tag{7}$$

where ϕ_0 and $\underset{\sim}{\phi}$ satisfy the harmonic equations

$$\nabla^2\phi_0 = 0 , \qquad \nabla^2\underset{\sim}{\phi} = \underset{\sim}{0}$$

In spherical coordinates, (7) yield

$$\begin{aligned} q_r &= \frac{\partial}{\partial r}[\phi_0 + r \sin\theta \, (\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 r \cos\theta] \\ &\quad - 2[\sin\theta \, (\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 \cos\theta] \\ q_\theta &= \frac{\partial}{r\partial\theta}[\phi_0 + r \sin\theta \, (\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 r \cos\theta] \\ &\quad - 2[\cos\theta \, (\phi_1 \cos\varphi + \phi_2 \sin\varphi) - \phi_3 \sin\theta] \\ q_\varphi &= \frac{1}{r \sin\theta} \frac{\partial}{\partial\varphi}[\phi_0 + r \sin\theta \, (\phi_1 \cos\varphi + \phi_2 \sin\varphi) + \phi_3 r \cos\theta] \\ &\quad - 2(\phi_2 \cos\varphi - \phi_1 \sin\varphi) \end{aligned} \tag{8}$$

Guided by the nature of $\phi^{(o)}$ and equations (8), we observe that $\phi_0 \propto \cos 2\varphi P_2^2$, $\phi_1 \propto \cos\varphi P_1^1$, $\phi_2 \propto -\sin\varphi P_1^1$ and $\phi_3 = 0$ since $P_1^1 = \sin\theta$.

And so for the region external to the drop we select our ϕ's as follows:

$$\phi_0 = \frac{A\cos 2\varphi P_2^2}{r^3}, \quad \phi_1 = \frac{3B\cos\varphi P_1^1}{r^2}, \quad \phi_2 = \frac{-3B\sin\varphi P_1^1}{r^2}, \quad \phi_3 = 0 \tag{9}$$

while for the interior of the drop we take

$$\phi_0 = C\, r^2 \cos 2\varphi P_2^2, \quad \phi_1 = \phi_2 = \phi_3 = 0. \tag{10}$$

where A, B and C are constants to be determined.

Substituting (9) in (8) we have

$$q_r^* = \frac{-3\cos 2\varphi\, P_2^2}{r^2}\left(\frac{A}{r^2} + B\right) \tag{11}$$

$$q_\theta^* = \frac{A\cos 2\varphi}{r^4}\frac{\partial P_2^2}{\partial\theta} \tag{12}$$

$$q_\varphi^* = \frac{-2A\sin 2\varphi P_2^2}{r^4 \sin\theta} \tag{13}$$

The tangential stress components $\sigma_{r\theta}^*$ and $\sigma_{r\varphi}^*$ are also found to be

$$\sigma_{r\theta}^* = \mu_1\left(\frac{\partial q_r^*}{r\partial\theta} + \frac{\partial q_\theta^*}{\partial r} - \frac{q_\theta^*}{r}\right)$$

$$= \frac{-\mu_1\cos 2\varphi}{r^3}\frac{\partial P_2^2}{\partial\theta}\left(\frac{8A}{r^2} + 3B\right) \tag{14}$$

and

$$\sigma_{r\varphi}^* = \mu_1\left(\frac{\partial q_r^*}{r\sin\theta\partial\varphi} + \frac{\partial q_\varphi^*}{\partial r} - \frac{q_\varphi^*}{r}\right)$$

$$= \frac{2\mu_1\sin 2\varphi P_2^2}{r^3\sin\theta}\left(\frac{8A}{r^2} + 3B\right) \tag{15}$$

so that external to the drop we have

$$q_r^{(1)} = q_r^{(o)} + q_r^{*}$$

$$= \cos 2\varphi P_2^2 \left[Kr - \frac{3}{r^2}\left(\frac{A}{r^2} + B\right)\right] \tag{16}$$

$$q_\theta^{(1)} = \cos 2\varphi \frac{\partial P_2^2}{\partial \theta}\left(\frac{Kr}{2} + \frac{A}{r^4}\right) \tag{17}$$

$$q_\varphi^{(1)} = -\frac{\sin 2\varphi P_2^2}{\sin \theta}\left(Kr + \frac{2A}{r^4}\right) \tag{18}$$

And since $$\sigma_{r\theta}^{(o)} = \mu_1 \left(\frac{\partial q_r^{(o)}}{r\partial\theta} + \frac{\partial q_\theta^{(o)}}{\partial r} - \frac{q_\theta^{(o)}}{r}\right)$$

$$= K \mu_1 \cos 2\varphi \frac{\partial P_2^2}{\partial \theta}$$

while $$\sigma_{r\varphi}^{(o)} = \mu_1 \left(\frac{\partial q_r^{(o)}}{r \sin \theta \partial\varphi} + \frac{\partial q_\varphi^{(o)}}{\partial r} - \frac{q_\varphi^{(o)}}{r}\right)$$

$$= - \frac{2K\mu_1 \sin 2\varphi P_2^2}{\sin \theta}$$

we have $$\sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(o)} + \sigma_{r\theta}^{*}$$

$$= \mu_1 \cos 2\varphi \frac{\partial P_2^2}{\partial \theta}\left[K - \frac{1}{r^3}\left(\frac{8A}{r^2} + 3B\right)\right] \tag{19}$$

and $$\sigma_{r\varphi}^{(1)} = -\frac{2\mu_1 \sin 2\varphi P_2^2}{\sin\theta}\left[K - \frac{1}{r^3}\left(\frac{8A}{r^2} + 3B\right)\right] \tag{20}$$

By substituting (10) into (8), we get for the interior of the drop

$$q_r^{(2)} = 2\ C\ r\ \cos\ 2\varphi P_2^2 \tag{21}$$

$$q_\theta^{(2)} = C\ r\ \cos\ 2\varphi\ \frac{\partial P_2^2}{\partial \theta} \tag{22}$$

$$q_\varphi^{(2)} = -\frac{2\ C\ r\ \sin\ 2\varphi P_2^2}{\sin\ \theta} \tag{23}$$

$$\sigma_{r\theta}^{(2)} = 2\mu_2\ C\ \cos\ 2\varphi\ \frac{\partial P_2^2}{\partial \theta} \tag{24}$$

$$\sigma_{r\varphi}^{(2)} = -\frac{4\ C\ \mu_2\ \sin\ 2\varphi P_2^2}{\sin\ \theta} \tag{25}$$

The boundary conditions to be satisfied are the continuity of velocity components and tangential stresses, that is, at $r = a$

$$\begin{aligned} q_r^{(1)} &= q_r^{(2)} \\ q_\theta^{(1)} &= q_\theta^{(2)} \\ q_\varphi^{(1)} &= q_\varphi^{(2)} \\ \sigma_{r\theta}^{(1)} &= \sigma_{r\theta}^{(2)} \\ \sigma_{r\varphi}^{(1)} &= \sigma_{r\varphi}^{(2)} \end{aligned} \tag{26}$$

where a is the radius of the drop.

Applying these boundary conditions yields the following equations:

$$Ka - \frac{3}{a^2}\left(\frac{A}{a^3} + B\right) = 2Ca \tag{27}$$

$$\frac{Ka}{2} + \frac{A}{a^4} = Ca \tag{28}$$

$$KA + \frac{2A}{a^4} = 2Ca \tag{29}$$

$$\mu_1\left[K - \frac{1}{a^3}\left(\frac{8A}{a^2} + 3B\right)\right] = 2\mu_2 C \quad . \tag{30}$$

$$\mu_1\left[K - \frac{1}{a^3}\left(\frac{8A}{a^2} + 3B\right)\right] = 2\mu_2 C \quad . \tag{31}$$

These equations are equivalent to

$$Ka^3 - 3\left(\frac{A}{a^2} + B\right) = 2\ Ca^3 \tag{32}$$

$$Ka^5 + 2A = 2\ Ca^5 \tag{33}$$

$$\left[Ka^3 - \left(\frac{8A}{a^2} + 3B\right)\right] = 2\ Cg\ a^3 \tag{34}$$

where $g = \mu_2/\mu_1$.

Solving these equations simultaneously yield

$$A = a^5 k g_0 \tag{35}$$

where $g_0 = \dfrac{1 - g}{2g + 3}$

$$B = -\frac{5a^3\ kg_0}{3} \tag{36}$$

and

$$C = \frac{5kg_1}{2} \tag{37}$$

where $g_1 = \dfrac{1}{2g + 3}$.

$\therefore$ Writing $q_r^{(o)}(r,\theta,\varphi) = k\ r\ \cos 2\varphi P_2^2$

then $q_r^{(o)}(R,\theta,\varphi) = k\ R\ \cos 2\varphi P_2^2$, where $R = \dfrac{a^2}{r^2}$

$$= k\ \frac{a^2}{r^2}\ \cos 2\varphi P_2^2$$

and $\quad q_r^{(o)}(R^2, \theta,\varphi) \quad = \quad k\,\frac{a^4}{r^4}\cos 2\varphi P_2^2\ .$

Then we can write

$$q_r^{(1)} = q_r^{(o)} + q_r^{*}$$

$$= q_r^{(o)} - \frac{3\cos 2\varphi P_2^2}{r^2}\left(\frac{A}{r^2} + B\right)$$

$$= q_r^{(o)} - \frac{3a^3kg_o\cos 2\varphi P_2^2}{r^2}\left(\frac{a^2}{r^2} - \frac{5}{3}\right)$$

$$= q_r^{(o)}(r,\theta,\varphi) + 5ag_o\, q_r^{(o)}(R,\theta,\varphi) - 3ag_o\, q_r^{(o)}(R^2,\theta,\varphi) \quad (38)$$

Similarly

$$q_\theta^{(1)} = q_\theta^{(o)}(r,\theta,\varphi) + 2ag_o\, q_\theta^{(o)}(R^2,\theta,\varphi) \quad (39)$$

$$q_\varphi^{(1)} = q_\varphi^{(o)}(r,\theta,\varphi) + 2ag_o\, q_\varphi^{(o)}(R^2,\theta,\varphi) \quad (40)$$

$$\sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(o)}(r,\theta,\varphi) - \frac{ag_o}{r}\left[5\int_o^R \sigma_{r\theta}^{(o)}(\lambda,\theta,\varphi)d\lambda + 8\int_o^{R^2} \sigma_{r\theta}^{(o)}(\lambda,\theta,\varphi)d\lambda\right] \quad (41)$$

and

$$\sigma_{r\varphi}^{(1)} = \sigma_{r\varphi}^{(o)}(r,\theta,\varphi) - \frac{a\,g_o}{r}\left[5\int_o^R \sigma_{r\varphi}^{(o)}(\lambda,\theta,\varphi)d\lambda + 8\int_o^{R^2} \sigma_{r\varphi}^{(o)}(\lambda,\theta,\varphi)d\lambda\right] \quad (42)$$

Proceeding in a similar manner we found that the following relationship also exists between the flow field within the drop and that of the homogeneous fluid:

$$q_r^{(2)} = 5g_1 \, q_r^{(o)} \tag{43}$$

$$q_\theta^{(2)} = 5g_1 \, q_\theta^{(o)} \tag{44}$$

$$q_\varphi^{(2)} = 5g_1 \, q_\varphi^{(o)} \tag{45}$$

$$\sigma_{r\theta}^{(2)} = M\sigma_{r\theta}^{(o)} \tag{46}$$

where $M = 5gg_1$

and $\sigma_{r\varphi}^{(2)} = M \, \sigma_{r\varphi}^{(o)}$ (47)

$\sigma_{rr}^{(1)}$ is found to be

$$\begin{aligned} \sigma_{rr}^{(1)} &= \sigma_{rr}^{(o)} + \frac{12\mu_1 \cos 2\varphi P_2^2}{r^3}\left(\frac{2A}{r^2} + B\right) \\ &= \sigma_{rr}^{(o)}(r,\theta,\varphi) - \frac{6ag_o}{r}\left[5\int_o^R \sigma_{rr}^{(o)}(\lambda,\theta,\varphi)d\lambda \right. \\ &\qquad \left. - 2\int_o^{R^2} \sigma_{rr}^{(o)}(\lambda,\theta,\varphi)d\lambda\right] \end{aligned} \tag{48}$$

and $\sigma_{rr}^{(2)} = 4 \, C \, \mu_2 \cos 2\varphi P_2^2 = 5gg_1 \, \sigma_{rr}^{(o)}(r,\theta,\varphi)$ (49)

so that we can write

$$\underset{\sim}{q}^{(2)} = 5g_1 \underset{\sim}{q}^{(o)}, \quad \underset{\sim}{\sigma}^{(2)} = 5gg_1 \underset{\sim}{\sigma}^{(o)} \tag{50}$$

$\therefore$ The force trying to deform the spherical drop is $F = \sigma_{rr}^{(1)} - \sigma_{rr}^{(2)}$ and is given by

$$F = (1-5gg_1)\,\sigma_{rr}^{(o)}(r,\theta,\varphi) - \frac{6ag_0}{r}\left[5\int_0^R \sigma_{rr}^{(o)}(\lambda,\theta,\varphi)d\lambda - 2\int_0^{R^2} \sigma_{rr}^{(o)}(\lambda,\theta,\varphi)d\lambda\right] \qquad (51)$$

On the surface of the sphere, i.e. at $r = a$,

$$F = (1-5gg_1)\,\sigma_{rr}^{(o)}(a,\theta,\varphi) - 6g_0\left[5\sigma_{rr}^{(o)}(a,\theta,\varphi) - 2\sigma_{rr}^{(o)}(a,\theta,\varphi)\right]$$

$$= (1-5gg_1 - 18g_0)\,\sigma_{rr}^{(o)}(a,\theta,\varphi) = -15g_0\sigma_{rr}^{(o)}(a;\theta,\varphi)$$

$$= -30g_0k\mu_1 \cos 2\varphi P_2^2 \qquad (52)$$

and is independent of the radius of the drop.

5.4 A SPECIAL CASE

If the spherical drop contains a fluid whose viscosity is very small compared to that of the surrounding fluid then

$\therefore\ \mu_2 \ll \mu_1$ and $g = \dfrac{\mu_2}{\mu_1}$ becomes negligible. Consequently,

$$g_0 = \frac{1-g}{2g+3} \doteqdot \frac{1}{3} \qquad (53)$$

and

$$g_1 = \frac{1}{2g+3} \doteqdot \frac{1}{3} \qquad (54)$$

$\therefore$ From (38), $q_r^{(1)}$ reduces to $q_r^{(3)}$ say, where

$$q_r^{(3)} = q_r^{(o)}(r,\theta,\varphi) + a\left(\frac{5}{3}\,q_r^{(o)}(R,\theta,\varphi) - q_r^{(o)}(R^2,\theta,\varphi)\right) \qquad (55)$$

Similarly $q_\theta^{(1)}$, $q_\varphi^{(1)}$, $\sigma_{r\theta}^{(1)}$, $\sigma_{r\varphi}^{(1)}$ and $\sigma_{rr}^{(1)}$ reduce respectively to $q_\theta^{(3)}$, $q_\varphi^{(3)}$, $\sigma_{r\theta}^{(3)}$, $\sigma_{r\varphi}^{(3)}$ and $\sigma_{rr}^{(3)}$ where

$$q_\theta^{(3)} = q_\theta^{(o)}(r,\theta,\varphi) + \frac{2a}{3}\, q_\theta^{(o)}(R^2,\theta,\varphi) \tag{56}$$

$$q_\varphi^{(3)} = q_\varphi^{(o)}(r,\theta,\varphi) + \frac{2a}{3}\, q_\theta^{(o)}(R^2,\theta,\varphi) \tag{57}$$

$$\sigma_{r\theta}^{(3)} = \sigma_{r\theta}^{(o)}(r,\theta,\varphi) - \frac{a}{3r}\left(5\int_0^R \sigma_{r\theta}^{(o)}(\lambda,\theta,\varphi)d\lambda + 8\int_0^{R^2} \sigma_{r\theta}^{(o)}(\lambda,\theta,\varphi)d\lambda\right) \tag{58}$$

$$\sigma_{r\varphi}^{(3)} = \sigma_{r\varphi}^{(o)}(r,\theta,\varphi) - \frac{a}{3r}\left(5\int_0^R \sigma_{r\varphi}^{(o)}(\lambda,\theta,\varphi)d\lambda + 8\int_0^{R^2} \sigma_{r\varphi}^{(o)}(\lambda,\theta,\varphi)d\lambda\right) \tag{59}$$

and

$$\sigma_{rr}^{(3)} = \sigma_{rr}^{(o)}(r,\theta,\varphi) - \frac{2a}{r}\left(5\int_0^R \sigma_{rr}^{(o)}(\lambda,\theta,\varphi)d\lambda - 2\int_0^{R^2} \sigma_{rr}^{(o)}(\lambda,\theta,\varphi)d\lambda\right) \tag{60}$$

Similarly $\underset{\sim}{g}^{(2)}$ reduces to $\underset{\sim}{g}^{(4)}$ say where

$$\underset{\sim}{g}^{(4)} = \frac{5}{3}\,\underset{\sim}{g}^{(o)} \tag{61}$$

and $\underset{\sim}{\sigma}^{(2)}$ vanishes.

The force of deformation acting on the spherical bubble is then given by $\sigma_{rr}^{(3)}$ which on the surface of the bubble has the value

$$J = \sigma_{rr}^{(o)}(a,\theta,\varphi) - 2(5\,\sigma_{rr}^{(o)}(\lambda,\theta,\varphi) - 2\,\sigma_{rr}^{(o)}(\lambda,\theta,\varphi))$$

$$= 5\,\sigma_{rr}^{(o)}(a,\theta,\varphi) = -10\,k\,\mu_1\,\cos 2\varphi P_2^2 \tag{62}$$

which again is independent of the radius of the bubble.

CHAPTER SIX

This chapter contains a summary of the conclusions reached in this thesis as well as a few deductions that could be made from the results obtained so far.

In chapter two, a method is developed which simplifies the integration of the governing steady state equations of an incompressible viscous fluid with zero body and inertia forces. These equations being

$$\nabla p = \mu \nabla^2 \underset{\sim}{q}, \qquad \nabla \cdot \underset{\sim}{q} = 0, \tag{1}$$

a solution is found to be

$$\underset{\sim}{q} = \nabla(\phi_0 + \underset{\sim}{r} \cdot \underset{\sim}{\phi}) - 2\underset{\sim}{\phi}, \qquad p = 2\mu\,(\nabla \cdot \underset{\sim}{\phi}), \tag{2}$$

where ϕ_0 and $\underset{\sim}{\phi}$ satisfy the harmonic equations

$$\nabla^2 \phi_0 = 0, \quad \nabla^2 \underset{\sim}{\phi} = \underset{\sim}{0}; \quad \text{and } \underset{\sim}{\phi} = (\phi_1, \phi_2), \tag{3}$$

in two-dimensions and

$$\underset{\sim}{\phi} = (\phi_1, \phi_2, \phi_3) \text{ in three-dimensions.} \tag{4}$$

This method makes use of potential theory to reduce any steady state fluid problem to that of differential operators acting upon harmonic functions. This is in contrast to the conventional stream function approach which employs a fourth-order partial differential equation and which can be used to solve only three-dimensional problems which are axisymmetric.

By this new approach, not only axisymmetric flow problems but non-axisymmetric flow problems also could be attempted.

For axisymmetric flow problems, only two harmonic equations,

$$\nabla^2\phi_0 = 0 = \nabla^2\phi_3 \tag{5}$$

need be solved since ϕ_1 and ϕ_2 vanish for such flows. For any type of flow, we need to solve at most four harmonic equations for a three-dimensional problem and at most three for a two-dimensional problem.

Furthermore, the formulation for singularities in the interior of one of two immiscible fluids becomes straight forward.

Three examples are considered. In chapter three, the new approach is employed to develop a theory for the flow of a thin jet in an infinite viscous fluid in steady state. It is found that the flow field can be described by a potential function

$$\phi = \sum_{n=0}^{\infty} L\, r^n P_n(\cos\theta) \tag{6}$$

where

$$L = -F/4\pi\mu h^{n+1}, \tag{7}$$

$P_n(\cos\theta)$ is the Legendre polynomial, F is the constant rate of change of the momentum of the jet and μ is the coefficient of viscosity of the infinite fluid.

In chapter four an axisymmetric flow problem is considered. It is the case of a source in an infinite fluid containing a spherical drop of another fluid. We found that the velocities and stresses within and without the spherical drop can be obtained by differentiations and integrations of the velocities and stresses of an infinite homogeneous fluid containing a source.

Furthermore, we found that the flow inside the drop could be described by a harmonic function

$$\phi_0^{(2)} = C\phi^{(0)}, \quad \text{where}$$

$$\phi^{(0)} = Gr^n P_n(\cos\theta), \quad \left(G = \frac{m}{n+1}\right), \quad n = 0, \ldots \infty. \tag{8}$$

is the potential function which describes the flow field in an homogeneous fluid. Also since

$$C = \frac{(2n-1)G}{1+2g(n-1)} = g^{-1}\left(1 + \frac{g-1}{2g(n+\alpha)}\right) \text{ where } \alpha = \frac{1-2g}{2g} \tag{9}$$

we have

$$\phi_0^{(2)} = g^{-1}\phi^{(0)} + \tfrac{1}{2}(g-1)g^{-2}r^{-\alpha}\int_0^r \lambda^{\alpha-1}\phi^{(0)}(\lambda,\theta)d\lambda. \tag{10}$$

Similarly, to describe the flow field exterior to the drop, two harmonic functions $\phi_0^{(1)}$ and $\phi_3^{(1)}$ are employed where $\phi_0^{(1)}$ and $\phi_3^{(1)}$ are

$$\phi_0^{(1)} = A\, r^{-(n+1)}P_n(\cos\theta) \tag{11}$$

and

$$\phi_3^{(1)} = \frac{B}{a^2}\, r^{-(n+1)}P_n(\cos\theta). \tag{12}$$

Now

$$A = \frac{n(n-1)(2n-1)(1-g)a^{2n+1}G}{2g(n+1)(n+\alpha)} = a^{2n+1}G(\alpha+\tfrac{1}{2})[-(5+2\alpha)+2n + \frac{\alpha(\alpha+1)(2\alpha+1)}{(\alpha-1)(n+\alpha)} - \frac{6}{(\alpha-1)(n+1)}] \tag{13}$$

$$\therefore \quad \phi_0^{(1)} = G\, R^n P_n \cdot \frac{a}{r}(\alpha+\tfrac{1}{2})[-(5+2\alpha)+2n + \frac{\beta}{n+\alpha} - \frac{6}{(\alpha-1)(n+1)}] \tag{14}$$

where $R = \frac{a^2}{r}$, $\beta = \frac{\alpha(\alpha+1)(2\alpha+1)}{\alpha-1}$

$$\therefore \quad \phi_0^{(1)}(r,\theta) = \frac{a}{r}(\alpha+\tfrac{1}{2})[-(5+2\alpha)\phi^{(0)}(R,\theta) + 2R\frac{\partial}{\partial R}\phi^{(0)}(R,\theta) + \beta R^{-\alpha}\int_0^R \lambda^{\alpha-1}\phi^{(0)}(\lambda,\theta)d\lambda - \frac{6R^{-1}}{\alpha-1}\int_0^R \phi^{(0)}(\lambda,\theta)d\lambda] \tag{15}$$

Similarly

$$B = \frac{(1-n)(2n+1)(1-g)a^{2n+1}G}{2g(n+1)(n+\alpha)} = (\alpha+\tfrac{1}{2})[-2+\frac{(\alpha+1)(2\alpha-1)}{(\alpha-1)(n+\alpha)} - \frac{2}{(\alpha-1)(n+1)}]a^{2n+1}G \tag{16}$$

So that

$$\phi_3^{(1)} = \frac{1}{ar}(\alpha+\tfrac{1}{2})GR^n(-2+\frac{\Upsilon}{n+\alpha} - \frac{2}{(\alpha-1)(n+1)}) \tag{17}$$

where

$$\Upsilon = \frac{(\alpha+1)(2\alpha-1)}{\alpha - 1}$$

$$\therefore \quad \phi_s^{(1)}(r,\theta) = \frac{1}{ar}(\alpha+\tfrac{1}{2})\Big[-2\phi^{(0)}(R,\theta) + \gamma\, R^{-\alpha}\int_0^R \lambda^{\alpha-1}\phi^{(0)}(\lambda,\theta)d\lambda - \frac{2R^{-1}}{\alpha-1}\int_0^R \phi^{(0)}(\lambda,\theta)d\lambda\Big]. \tag{18}$$

When the spherical drop is replaced by a gas bubble, we found that the flow fields inside and outside the spherical bubble are still directly deducible from the flow field of homogeneous fluid.

In chapter five, a non-axisymmetric flow problem is considered. This involves a spherical drop of fluid in an infinite dissimilar fluid undergoing shear flow. The results obtained showed clearly that the flow fields inside and outside the spherical drop of fluid are also directly deducible from that of homogeneous fluid in shear. This implies that what is true of an axisymmetric flow is also true for a non-axisymmetric flow.

Finally, if one considers also the fact that the conclusions reached in the case of the source problem are equally valid for a sink and a radial doublet, one begins to suspect that for a fairly wide range of flow problems the flow fields in two-fluid spaces might be deducible from that of a homogeneous fluid.

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