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ScienceDirect
Journal of the Nigerian Mathematical Society 34 (2015) 249–258
Journal of the Nigerian Mathematical Society
www.elsevier.com/locate/jnnms

On common fixed points and multipled fixed points of contractive mappings in metric-type spaces

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Received 12 July 2014; received in revised form 10 April 2015; accepted 28 April 2015
Available online 17 June 2015

Abstract

This research work entails the study of the existence of common fixed points of some Ciric classes of contractive mappings in cone b-metric spaces. The main result obtained unifies, improves and generalizes several results in literature including those of Abbas et al. (2010) and Huang and Xu (2012). Furthermore, as a way of applications, the result is used to discuss common coupled, tripled and multipled fixed points of contractive maps defined on cone b-metric spaces, via product cone b-metric spaces.

Keywords: Contractive maps; Fixed point theorems; Multipled fixed points; Cone b-metric spaces

1. Introduction

The Banach contraction principle proved by Banach [1] in a complete metric space was the starting point of exhaustive research in the fixed point theory. Many contractive conditions under which a map or set of maps have fixed points or common fixed points have been studied in metric spaces (see, for example, [2,3]). Generalized metric spaces have also been considered with the introduction of b-metric spaces [4], cone metric spaces [5] and recently, cone b-metric spaces [6]. Recall that a b-metric defined on a nonempty set X is a symmetric function $d : X \times X \to \mathbb{R}_+$ that satisfies the identity of indiscernibles (or coincidence axiom) and a distorted triangle inequality $d(x, z) \leq K[d(x, y) + d(y, z)]\ \forall x, y, z \in X$, where K is a fixed constant greater or equal to 1.

The results in Abbas et al. [7] and Olaleru and Olaoluwa [8] are a comprehensive generalization of many previous works on contractive mappings in cone metric spaces [9,10]. They established conditions under which four maps tied by a contractive condition have a common fixed point.

Huang and Xu [11] presented some new examples in cone b-metric spaces and proved some fixed point theorems of contractive mappings without the assumption of normality in cone b-metric spaces. In this paper, we generalize the results of Abbas et al. [7] to the context of cone b-metric spaces. Furthermore, the use of functions instead of constants in the contractive conditions studied improves and unifies most results, along this research interest, in literature.

The following definitions and results will be needed in the sequel.

Peer review under responsibility of Nigerian Mathematical Society.
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http://dx.doi.org/10.1016/j.jnnms.2015.06.001
0189-8965/

Definition 1 (*See [5]*)**.** Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non-empty and $P \neq \{0\}$;
(b) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

Given a cone P, define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in int\, P$, where $int\, P$ stands for interior of P. Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P in a normed space E is called normal whenever there is a real number $k > 0$, such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P.

In the following, we always suppose that E is a Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2 (*See [5]*)**.** Let X be a non-empty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \longrightarrow E$ satisfies:

(c_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and ony if $x = y$;
(c_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 3 (*See [6]*)**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and ony if $x = y$;
(b_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a cone b-metric space.

Obviously, cone b-metric spaces generalize b-metric spaces and cone metric spaces. Here are some examples:

Example 4. Let $X = \{1, 2, \ldots, n\}$; $E = \mathbb{R}^2$; $P = \{(x, y) \in E : x \geq 0, y \leq 0\}$. Define $d : X \times X \to E$ by

$$d(x, y) = \begin{cases} \left(\dfrac{1}{|x - y|}, -|x - y|\right) & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

If $n \notin \{2, 3\}$, then (X, d) is a cone b-metric space with the coefficient $s = \frac{(n-1)(n-2)}{2n-3} > 1$ and not a cone metric space since the triangle inequality fails for the points $1, 2, n$. If $n \in \{2, 3\}$, then (X, d) is a cone metric space.

Example 5 (*See [6]*)**.** Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d : X \times X \to \mathbb{R}_+$ be defined by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$. Then (X, d) is a b-metric space. Put $E = l^1$, $P = \{\{x_n\} \in E : x_n \geq 0, \ \forall n \geq 1\}$. Letting $\bar{d} : X \times X \to E$ be defined by $\bar{d}(x, y) = \left\{\frac{d(x,y)}{2^n}\right\}_{n \geq 1}$, $(X, \bar{d})$ is a cone b-metric space with the coefficient $s = 2^{\frac{1}{p}} > 1$ but it is not a cone metric space.

Definition 6 (*See [6]*)**.** Let (X, d) be a cone b-metric space, $\{x_n\}$ a sequence in X and $x \in X$. We say that $\{x_n\}$ is

- a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $d(x_n, x_m) \ll c$;
- a convergent sequence if for every $c \in E$ with $0 \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n \geq k$, $d(x_n, x) \ll c$. Such x is called limit of the sequence $\{x_n\}$.

Note that every convergent sequence in a cone b-metric space X is a Cauchy sequence. A cone b-metric space X is said to be complete if every Cauchy sequence in X is convergent in X. The following lemma will be needed in the sequel:

Lemma 7. *Let (X, d) be a cone b-metric space X with the coefficient $s \geq 1$. Suppose that the sequence $\{y_n\} \subset X$ be such that there is $\lambda \in [0, \frac{1}{s})$ such that $d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$ for each $n \geq 1$. Then $\{y_n\}$ is Cauchy.*

Proof. For $n \in \mathbb{N}$, we have $d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \leq \lambda^2 d(y_{n-2}, y_{n-1}) \leq \cdots \leq \lambda^n d(y_0, y_1)$. For any $n, p \in \mathbb{N}$, we have:

$$\begin{aligned}
d(y_n, y_{n+p}) &\leq s[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+p})] \\
&= sd(y_n, y_{n+1}) + sd(y_{n+1}, y_{n+p}) \\
&\leq sd(y_n, y_{n+1}) + s^2[d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+p})] \\
&= sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + s^2 d(y_{n+2}, y_{n+p}) \\
&\leq \\
&\vdots \\
&\leq sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + \cdots + s^{p-1} d(y_{n+p-2}, y_{n+p-1}) + s^{p-1} d(y_{n+p-1}, y_{n+p}) \\
&\leq [s\lambda^n + s^2\lambda^{n+1} + s^3\lambda^{n+2} + \cdots + s^{p-1}\lambda^{n+p-2} + s^{p-1}\lambda^{n+p-1}] d(y_0, y_1) \\
&\leq s\lambda^n \sum_{k=0}^{p-1} (s\lambda)^k d(y_0, y_1) = s\lambda^n \frac{1}{1-(s\lambda)} d(y_0, y_1).
\end{aligned}$$

Given $0 \ll c$, choose $\tau > 0$ such that $c + \{y \in P : y < \tau\} \subset P$. Since $s\lambda^n \frac{1}{1-s\lambda} \to 0$ as $n \to \infty$, there is $n_0 \in \mathbb{N}$ such that $s\lambda^n \frac{1}{1-s\lambda} d(y_0, y_1) \in \{y \in P : y < \tau\}$ for all $n > n_0$. It follows that $s\lambda^n \frac{1}{1-s\lambda} d(y_0, y_1) \ll c$ for all $n > n_0$. Thus for all $n > n_0$ and $p \in \mathbb{N}$, $d(y_n, y_{n+p}) \ll c$ and $\{y_n\}$ is Cauchy. $\square$

Definition 8 (*See [12,13]*)**.** Let X be a set and let f, g be two self-mappings of X.

(i) A point $x \in X$ is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.
(ii) f and g are weakly compatible (w-compatible) if they commute at all their coincidence points.
(iii) f and g are occasionally weakly compatible (owc) iff there is a point $x \in X$ which is a coincidence point of f and g at which f and g commute.

It should be noted that the concept of occasionally weak compatibility is a proper generalization of nontrivial weak compatibility for maps which do have a coincidence point. However, if two occasionally weakly compatible maps f and g have just one point of coincidence (even with many coincidence points), then they are weakly compatible: If $x_1, x_2, \ldots, x_n$ are n coincidence points of f and g and w is the unique point of coincidence, then $w = fx_i = gx_i$ for all i; thus "f and g owc" implies that $fgx_{i_0} = gfx_{i_0}$ for one $i_0 \in \{1, 2, \ldots, n\}$, that is $fw = gw$ and $fgx_i = gfx_i$ for all $i \in \{1, 2, \ldots, n\}$.

2. Fixed points of contractive mappings in cone *b*-metric spaces

We begin this section by proving the existence of common fixed points for four contractive self maps of Ciric type (see [14]) in cone b-metric spaces using a methodology inspired by [14,7,6]. The following Theorem 9 generalizes all the results in [7] and the references therein to cone b-metric spaces. Furthermore, when $s = 1$, it unifies their results in the sense that the choice of functions, as in [14], instead of constants, as in [7], permits us to obtain their two theorems as corollaries.

Theorem 9. *Let f, g, S and T be self-mappings of a cone b-metric space X with the coefficient $s \geq 1$, satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$\begin{aligned}
d(fx, gy) \leq\ & a_1(x, y)d(Sx, Ty) + a_2(x, y)d(fx, Sx) + a_3(x, y)d(gy, Ty) \\
& + a_4(x, y)[d(fx, Ty) + d(gy, Sx)] \qquad (1)
\end{aligned}$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4 : X \times X \to [0, \frac{1}{s})$ satisfy

$$\sup_{x,y\in X} \{a_1(x, y) + a_2(x, y) + a_3(x, y) + 2sa_4(x, y)\} \le \lambda < \frac{1}{s}. \tag{2}$$

If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover if $\{f, S\}$ and $\{g, T\}$ are occasionally weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. Given that $f(X) \subset T(X)$ and $g(X) \subset S(X)$, and given $x_0 \in X$, one can define sequences $\{x_n\}$ and $\{y_n\}$ such that $y_{2n-1} := fx_{2n-2} = Tx_{2n-1}$ and $y_{2n} := gx_{2n-1} = Sx_{2n}$ for all $n \in \mathbb{N}$.

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n-1}) \\ &\le a_1(\alpha_n)d(Sx_{2n}, Tx_{2n-1}) + a_2(\alpha_n)d(fx_{2n}, Sx_{2n}) \\ &\quad + a_3(\alpha_n)d(gx_{2n-1}, Tx_{2n-1}) + a_4(\alpha_n)[d(fx_{2n}, Tx_{2n-1}) + d(gx_{2n-1}, Sx_{2n})] \\ &\le a_1(\alpha_n)d(y_{2n-1}, y_{2n}) + a_2(\alpha_n)d(y_{2n+1}, y_{2n}) + a_3(\alpha_n)d(y_{2n}, y_{2n-1}) + a_4(\alpha_n)d(y_{2n+1}, y_{2n-1}) \\ &\le a_1(\alpha_n)d(y_{2n-1}, y_{2n}) + a_2(\alpha_n)d(y_{2n+1}, y_{2n}) \\ &\quad + a_3(\alpha_n)d(y_{2n}, y_{2n-1}) + a_4(\alpha_n)s[d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n-1})] \\ &\le [a_1(\alpha_n) + a_3(x_{2n}, x_{2n-1}) + sa_4(\alpha_n)]d(y_{2n-1}, y_{2n}) + [a_1(\alpha_n) + sa_4(\alpha_n)]d(y_{2n}, y_{2n+1}), \end{aligned}$$

where $\alpha_n = (x_{2n}, x_{2n-1})$.

Hence $d(y_{2n}, y_{2n+1}) \le \delta(x_{2n}, x_{2n-1})d(y_{2n-1}, y_{2n})$ where $\delta(x, y) = \frac{a_1(x,y)+a_3(x,y)+s\cdot a_4(x,y)}{1-a_2(x,y)-s\cdot a_4(x,y)}$.

Since $\lambda < \frac{1}{s} \le 1$, from $a_1(x, y) + \lambda a_2(x, y) + a_3(x, y) + \lambda sa_4(x, y) + sa_4(x, y) \le \lambda$, we have $\delta(x, y) \le \lambda$; hence for all $n \in \mathbb{N}$

$$d(y_{2n}, y_{2n+1}) \le \lambda d(y_{2n-1}, y_{2n}). \tag{3}$$

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\le a_1(\beta_n)d(Sx_{2n}, Tx_{2n+1}) + a_2(\beta_n)d(fx_{2n}, Sx_{2n}) \\ &\quad + a_3(\beta_n)d(gx_{2n+1}, Tx_{2n+1}) + a_4(\beta_n)[d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})] \\ &\le a_1(\beta_n)d(y_{2n+1}, y_{2n+2}) + a_2(\beta_n)d(y_{2n+1}, y_{2n+2}) \\ &\quad + a_3(\beta_n)d(y_{2n+1}, y_{2n+2}) + a_4(\beta_n)d(y_{2n}, y_{2n+2}) \\ &\le [a_1(\beta_n) + a_2(\beta_n) + a_3(\beta_n)]d(y_{2n+1}, y_{2n+2}) + a_4(\beta_n)s[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] \\ &\le [a_1(\beta_n) + a_2(\beta_n) + a_3(\beta_n) + sa_4(\beta_n)]d(y_{2n+1}, y_{2n+2}) + sa_4(\beta_n)d(y_{2n}, y_{2n+1}), \end{aligned}$$

where $\beta_n = (x_{2n}, x_{2n+1})$.

Hence $d(y_{2n+1}, y_{2n+2}) \le \delta'(x_{2n}, x_{2n+1})d(y_{2n}, y_{2n+1})$ where $\delta'(x, y) = \frac{sa_4(x,y)}{1-a_1(x,y)-a_2(x,y)-a_3(x,y)-sa_4(x,y)}$.

Since $\lambda < \frac{1}{s} \le 1$, from $\lambda[a_1(x, y) + a_2(x, y) + a_3(x, y) + s \cdot a_4(x, y)] + sa_4(x, y) \le \lambda$, we have $\delta'(x, y) \le \lambda$; hence for all $n \in \mathbb{N}$

$$d(y_{2n+1}, y_{2n+2}) \le \lambda d(y_{2n}, y_{2n+1}). \tag{4}$$

From (3) and (4) we have that $d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n)$ for all $n \ge 2$.

From Lemma 7, $\{y_n\}$ is a Caucy sequence.

Suppose that $S(X)$ is complete. Then there exists $u \in S(X)$, say $u = Sv$, such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$. In fact, $y_n \to u$ as $n \to \infty$. Let us prove that $fv = u$.

$$\begin{aligned} d(fv, gx_{2n-1}) &\le a_1(\gamma_n)d(Sv, Tx_{2n-1}) + a_2(\gamma_n)d(fv, Sv) + a_3(\gamma_n)d(gx_{2n-1}, Tx_{2n-1}) \\ &\quad + a_4(\gamma_n)[d(fv, Tx_{2n-1}) + d(gx_{2n-1}, Sv)] \\ &\le a_1(\gamma_n)d(u, y_{2n-1}) + a_2(\gamma_n)d(fv, u) + a_3(\gamma_n)d(y_{2n}, y_{2n-1}) \\ &\quad + a_4(\gamma_n)[sd(fv, u) + sd(u, y_{2n-1}) + d(y_{2n}, u)] \\ &\le [a_1(\gamma_n) + sa_4(\gamma_n)]d(u, y_{2n-1}) + [a_2(\gamma_n) + sa_4(\gamma_n)]d(fv, u) \\ &\quad + a_3(\gamma_n)d(y_{2n}, y_{2n-1}) + a_4(\gamma_n)d(y_{2n}, u) \\ &\le \lambda[d(u, y_{2n-1}) + d(fv, u) + d(y_{2n}, y_{2n-1}) + d(y_{2n}, u)], \end{aligned}$$

with $\gamma_n = (v, x_{2n-1})$. On taking $n \to \infty$, $d(fv, u) \leq \lambda d(fv, u)$. Since $\lambda < 1$, we have that $d(fv, u) = 0$, i.e. $fv = u$. Thus $u = Sv = fv$.

Since $u \in f(X) \subset T(X)$, there exists $w \in X$ such that $Tw = u$. Now we shall show that $gw = u$.

$$\begin{aligned} d(fx_{2n}, gw) &\leq a_1(\delta_n)d(Sx_{2n}, Tw) + a_2(\delta_n)d(fx_{2n}, Sx_{2n}) \\ &\quad + a_3(\delta_n)d(gw, Tw) + a_4(\delta_n)[d(fx_{2n}, Tw) + d(gw, y_{2n})] \\ &\leq a_1(\delta_n)d(y_{2n}, u) + a_2(\delta_n)d(y_{2n+1}, y_{2n}) + a_3(\delta_n)d(gw, Tw) \\ &\quad + a_4(\delta_n)[d(y_{2n+1}, Tw) + sd(gw, Tw) + sd(Tw, y_{2n})] \\ &\leq [a_1(\delta_n) + sa_4(\delta_n)]d(y_{2n}, u) + a_2(\delta_n)d(y_{2n+1}, y_{2n}) \\ &\quad + [a_3(\delta_n) + sa_4(\delta_n)]d(gw, Tw) + a_4(\delta_n)[d(y_{2n+1}, u)] \\ &\leq \lambda[d(y_{2n}, u) + d(y_{2n+1}, y_{2n}) + d(gw, u) + d(y_{2n+1}, u)], \end{aligned}$$

where $\delta_n = (x_{2n}, w)$. On taking $n \to \infty$, $d(gw, u) \leq \lambda d(gw, u)$, i.e., $gw = u$. Thus $u = gw = Tw$.

Suppose that there exists another coincidence point v^* of the pair $\{f, S\}$. The contractive condition (1) yields:

$$\begin{aligned} d(fv^*, gw) &\leq a_1(v^*, w)d(Sv^*, Tw) + a_2(v^*, w)d(fv^*, Sw) + a_3(v^*, w)d(gw, Tw) \\ &\quad + a_4(v^*, w)[d(fv^*, Tw) + d(gw, Sv^*)] \end{aligned} \tag{5}$$

i.e. $d(fv^*, fv) \leq a_1(v^*, w)d(fv^*, fv) + a_4(v^*, w)[d(fv^*, fv) + d(fv, fv^*)] \leq \lambda d(fv^*, fv)$. Since $\lambda < 1$, $d(fv^*, fv) = 0$, and so $u = fv = Sv = fv^* = Sv^*$ is the unique point of coincidence of $\{f, S\}$.

Similarly $u = gw = Tw$ is the unique point of coincidence of $\{g, T\}$.

If the pairs $\{f, S\}$ and $\{g, T\}$ are occasionally weakly compatible, then for some coincidence point v of $\{f, S\}$ and w of $\{g, T\}$, $fu = fSv = Sfv = Su = w_1$ (say) and $gu = gTw = Tgw = Tu = w_2$ (say).

$$\begin{aligned} d(w_1, w_2) = d(fu, gu) &\leq a_1(u, u)d(Su, Tu) + a_2(u, u)d(fu, Su) \\ &\quad + a_3(u, u)d(gu, Tu) + a_4(u, u)[d(fu, Tu) + d(gu, Su)] \\ &= [a_1(u, u) + 2a_4(u, u)]d(w_1, w_2) \leq \lambda d(w_1, w_2) \end{aligned}$$

which implies that $w_1 = w_2$. Therefore $fu = gu = Su = Tu$. Now we show that $u = gu$.

$$\begin{aligned} d(u, gu) = d(fv, gu) &\leq a_1(v, u)d(Sv, Tu) + a_2(v, u)d(fv, Sv) \\ &\quad + a_3(v, u)d(gu, Tu) + a_4(v, u)[d(fv, Tu) + d(gu, Sv)] \\ &= [a_1(v, u) + 2a_4(v, u)]d(gu, u) \leq \lambda d(gu, u) \end{aligned}$$

and $gu = u$. Thus u is a common fixed point of f, g, S and T. The uniqueness of the common fixed point is an immediate consequence of the generalized contractive condition. The same can be proved if we assume instead that $T(X)$, $f(X)$ or $g(X)$ is complete. □

When the functions a_1, a_2, a_3, a_4 in the previous theorem are constants, we have the following corollary:

Corollary 10. *Let f, g, S and T be self-mappings of a cone b-metric space X with the coefficient $s \geq 1$, satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$d(fx, gy) \leq a_1 d(Sx, Ty) + a_2 d(fx, Sx) + a_3 d(gy, Ty) + a_4[d(fx, Ty) + d(gy, Sx)]$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4 \in [0, \frac{1}{s})$ satisfy $a_1 + a_2 + a_3 + 2sa_4 = \lambda < \frac{1}{s}$.

If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover if $\{f, S\}$ and $\{g, T\}$ are (occasionally) weakly compatible, then f, g, S and T have a unique common fixed point.

Corollary 11. *Let f, g, S and T be self-maps of a cone metric space X with cone P having non-empty interior, satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$d(fx, gy) \leq hu_{x,y}(f, g, S, T), \tag{6}$$

where $h \in (0, \frac{1}{s})$ and

$$u_{x,y}(f, g, S, T) \in \left\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(fx, Ty) + d(gy, Sx)}{2s}\right\} \tag{7}$$

for all $x, y \in X$. If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence. Moreover, if $\{f, S\}$ and $\{g, T\}$ are (occasionally) weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. If

$$u_{x,y}(f, g, S, T) \in \left\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(fx, Ty) + d(gy, Sx)}{2s}\right\}$$

then

$$\begin{aligned} hu_{x,y}(f, g, S, T) &= a_1(x, y)d(Sx, Ty) + a_2(x, y)d(fx, Sx) \\ &\quad + a_3(x, y)d(gy, Ty) + a_4(x, y)[d(fx, Ty) + d(gy, Sx)] \end{aligned}$$

where $a_1, a_2, a_3 : X \times X \to \{0, h\}$, $a_4 : X \times X \to \{0, \frac{h}{2s}\}$ and $a_1(x, y) + a_2(x, y) + a_3(x, y) + 2sa_4(x, y) = h < \frac{1}{s}$. (This is possible when no two of a_1, a_2, a_3, a_4 can be simultaneously nonzero.)

It follows that f, g, S and T satisfy the contractive condition of Theorem 9, hence, they have a unique common fixed point. $\square$

Note that Corollaries 10 and 11 are generalizations of the main results in [7] to cone b-metric spaces: when $s = 1$ (cone metric spaces setting), Corollaries 10 and 11 become respectively Theorems 2.2 and 2.8 in [7]. Theorem 9 unifies the mentioned results then generalize them to cone b-metric spaces. Also note that the methodology of proof of Corollary 11 is different from the methodology of proof of Theorem 2.8 in [7], perhaps even simpler.

Example 12. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \{1, 2, \ldots, n\}$, $n > 3$, and $d : X \times X \to E$ defined by $d(x, y) = (\frac{1}{|x-y|}, \frac{1}{|x-y|})$. (X, d) is a cone b-metric space with coefficient $s = \frac{(n-1)(n-2)}{2n-3}$. Consider the maps $f, g, S, T : X \to X$ defined by $fx = Tx = \begin{cases} 1, & x = 1 \\ n, & x \neq 1 \end{cases}$, $g(x) = 1$ $\forall x \in X$ and $Sx = \begin{cases} 1, & x = 1 \\ 2, & x \neq 1 \end{cases}$. The conditions of Theorem 9 are satisfied with $a_1(x, y) = \frac{n-2}{n-1} < 1$ and $a_2(x, y) = a_3(x, y) = a_4(x, y) = 0$. 1 is the unique common fixed point of f, g, S, T.

3. Applications

3.1. Multipled fixed points and fixed points in product spaces

Following the definition of coupled (e.g. [15,16]), tripled (e.g. [17]) and even quadruple fixed points (see [18]), it is natural to generalize such notions to multipled fixed points. We recall here some definitions of the concept of multipled fixed points as stated by Olaoluwa and Olaleru [19] and which are improvements of the definitions introduced by Samet and Vetro [20] and Nashine et al. [21].

Let X be a nonempty set. Define, for any vector $x = (x_1, x_2, \ldots, x_m) \in X^m$, the circular matrix of x

$$t(x) := \begin{pmatrix} x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & x_m \\ x_2 & x_3 & \cdots & x_{m-1} & x_m & x_1 \\ x_3 & x_4 & \cdots & x_m & x_1 & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ x_m & x_1 & \cdots & x_{m-3} & x_{m-2} & x_{m-1} \end{pmatrix}.$$

In the sequel, $t_i(x)$ denotes the ith line of $t(x)$ and $t_{ij}(x)$ the (i, j)-th element of the matrix.

Definition 13 (*See [19]*)**.** Let X be a nonempty set and $F : X^m \to X$ and $g : X \to X$ be two mappings. An element $x = (x_1, x_2 \dots, x_m) \in X^m$ is said to be a coincidence point of m-order (or simply, multipled coincidence point) of F and g if

$$\begin{cases} F(x_1, x_2, \dots, x_{m-1}, x_m) = g(x_1) \\ F(x_2, x_3, \dots, x_m, x_1) = g(x_2) \\ F(x_3, x_4, \dots, x_m, x_1, x_2) = g(x_3) \\ \vdots \\ F(x_m, x_1, x_2, \dots, x_{m-1}) = g(x_m) \end{cases}$$

or equivalently $F(t_i(x)) = g(x_i),\ \forall i \in \{1, 2, \dots, m\}$.

In such case, $(gx_1, gx_2, \dots, gx_m)$ is said to be a multipled point of coincidence of F and g.

If in addition, all the x_i are fixed points of g, then x is said to be a common fixed point of m-order (or common multipled fixed point) of F and g.

If $g = Id_X$, then x is said to be a fixed point of m-order or multipled fixed point of F.

When $m = 1, 2, 3, 4$, we obtain the notions of fixed points, coupled fixed points, tripled fixed points and quadruple fixed points respectively.

Example 14 (*See [19]*). Let $X = \mathbb{R}$ and $F : X^m \to X$ be defined for all $x = (x_1, x_2, \dots, x_m)$ by $F(x) = 2x_1 + x_2 + x_3 + \cdots x_m - 1$. The system $F(t_i(x)) = x_i\ \forall i \in \{1, \dots, m\}$, is satisfied by all x such that $\sum_{j=1}^m x_j = 1$. In particular, $\left(\frac{1}{m}, \dots, \frac{1}{m}\right)$ and $(1, 0 \dots, 0)$ are both multipled fixed points of F.

Definition 15 (*See [19]*)**.** Let X be a nonempty set and $F : X^m \to X$ and $g : X \to X$ be two mappings. The mappings F and g are called

(w_1) w-compatible if $g(F(x_1, x_2, \dots, x_m)) = F(gx_1, gx_2, \dots, gx_m)$ at any multipled coincidence point $(x_1, x_2, \dots, x_m)$ of F and g.

(w_2) w*-compatible if $g(F(x, x, \dots, x)) = F(gx, gx, \dots, gx)$ whenever $gx = F(x, x, \dots, x)$.

The relationship between multipled fixed points and fixed points in product spaces is hereby established by considering what we tag "*associate mappings*" and proving the subsequent lemma. Consider the mappings $F : X^m \to X$ and $g : X \to X$ and the "*associate*" mappings $\tilde{F} : X^m \to X^m$ and $\tilde{g} : X^m \to X^m$ defined for all $x = (x_1, x_2, \dots, x_m) \in X^m$ by

$$\begin{cases} \tilde{F}(x) = (F(t_1(x)), F(t_2(x)), \dots, F(t_m(x))) \\ \tilde{g}(x) = (gx_1, gx_2, \dots, gx_m). \end{cases} \tag{8}$$

The following lemma is obtained.

Lemma 16 (*See [19]*)**.**

(i) *An element $x = (x_1, x_2, \dots, x_m) \in X^m$ is a multipled fixed point of F or multipled coincidence point (or common multipled fixed point) of F and g if and only if it is a fixed point of $\tilde{F}$ or multipled coincidence point (or common multipled fixed point) of $\tilde{F}$ and $\tilde{g}$.*

(ii) *The maps F and g are w-compatible if and only if $\tilde{F}$ and $\tilde{g}$ are w-compatible in X^m.*

The form of a multipled coincidence (or common fixed) point of F and g when it is unique and how it interrelates the concepts of w-compatibility and w*-compatibility is expressed in the following remark:

Remark 17 (*See [19]*)**.**

(i) If $x = t_1(x) = (x_1, x_2, \dots, x_m) \in X^m$ is a multipled coincidence point, multipled point of coincidence or common multipled fixed point of F and g then, by permutation, the elements $t_2(x), t_3(x), \dots, t_m(x)$ (where $t(x)$ is the circular matrix of x) are also multipled coincidence points, multipled points of coincidence or common multipled fixed points of F and g. Hence, if x is unique as multipled coincidence point, multipled point of coincidence or common multipled fixed point of F and g, then $x = t_1(x) = t_2(x) = \cdots = t_m(x)$, and so $x_1 = x_2 = \cdots = x_m$.

(ii) If F and g are w*-compatible mappings with only one multipled coincidence point, they are also w-compatible since, in such case, they would commute at their unique multipled coincidence point of the form $(x, x, \dots, x)$ from (i).

Example 18 (*See [19]*). Let $X = \mathbb{R}$ and $F : X^m \to X$ be defined by $F(x) = 1 - m + \sum_{j=1}^{m} x_j$. $F(t_i(x)) = x_i\ \forall i \in \{1, \dots, m\} \iff \sum_{j \neq i} x_j = m - 1$. The determinant of the system $\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = (-1)^{m-1}(m-1) \neq 0$, hence the system has a unique solution, $(1, \dots, 1)$, which is the unique multipled fixed point of F.

The notions of coupled fixed points, tripled fixed points and multipled fixed points in general are relative to mappings defined on X^m, with $m \geq 1$. The previous section discusses the existence of fixed points of contractive maps defined in a cone b-metric space X. It is therefore of interest to equip X^m with the same cone b-metric structure. The following subsection arises from this motivation and the notions therein introduced are generalizations of the notion of product cone metric spaces introduced by Olaoluwa and Olaleru [19].

3.2. Finite product cone b-metric spaces

Definition 19. Let (X_i, d_i), $i \in \{1, 2, \dots, m\}$ be m cone b-metric spaces with respect to cones P_i and coefficients s_i, where $P_i \subset E$ for all i and $P_i \cap (-P_j) = \{0\}$ for all i, j. The set $Z = \prod_{i=1}^{i=m} X_i$ together with $d : Z \times Z \to E$ defined by

$$d(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i), \quad \forall x = (x_1, x_2, \dots, x_m),\ y = (y_1, y_2, \dots, y_m)$$

is a cone b-metric space with respect to cone $P = \sum_{i=1}^{m} P_i$ and coefficient $s = \max s_i$. Z is called product cone b-metric space.

When $X_i = X$ for each $i \in \{1, 2, \dots, m\}$, where (X, d) is a cone b-metric space with respect to cone $P \subset E$, we define the product cone b-metric space X^m with respect to P by considering the cone b-metric $D : X^m \times X^m \to E$ by

$$D(x, y) = \sum_{i=1}^{m} d(x_i, y_i), \quad x = (x_i)_{1 \leq i \leq m},\ y = (y_i)_{1 \leq i \leq m}. \tag{9}$$

Note that when $s = 1$, the notion of product cone b metric space coincides with that of product cone metric space introduced by Olaoluwa and Olaleru [19]. Convergence of sequences in a product cone b-metric spaces and convergence of their coordinates are equivalent as expressed in the next proposition which is easy to prove.

Proposition 20. *Let (X, d) be a cone b-metric space and (X^m, D) the product cone b-metric space.*

(p_1) *A sequence $\{x_n\} = \{(x_n^1, x_n^2, \dots, x_n^m)\}$ converges to $x = (x^1, x^2, \dots, x^m)$ if and only if the sequences $\{x_n^i\}$ converge to x^i for all $i \in \{1, 2, \dots, m\}$.*
(p_2) *A sequence $\{x_n\} = \{(x_n^1, x_n^2, \dots, x_n^m)\}$ is a Cauchy sequence in X^m if and only if the sequences $\{x_n^i\}$ are Cauchy sequences for all $i \in \{1, 2, \dots, m\}$.*
(p_3) *(X^m, D) is complete if and only if (X, d) is complete.*

3.3. Consequences

Theorem 21. *Let (X, d) be a cone b-metric space with coefficient $s \geq 1$, $f : X^m \to X$, $g : X^m \to X$, $S : X \to X$ and $T : X \to X$ be four mappings such that $f(X^m) \subset T(X)$, $g(X^m) \subset S(X)$ and*

$$d(fx, gu) \leq \sum_{i=1}^{m} p_i d(Sx_i, Tu_i) + qd(fx, Sx_1) + rd(gu, Tu_1) + t[d(fx, Tu_1) + d(gu, Sx_1)] \tag{10}$$

for all $x = (x_1, x_2, \ldots, x_m), u = (u_1, u_2, \ldots, u_m) \in X^m$*, where* $p_i, (i = 1, 2, \ldots, m)$*,* $q, r, t \in (0, \frac{1}{s})$ *and* $\sum_{i=1}^m p_i + q + r + 2st < \frac{1}{s}$.

If one of $f(X^m)$*,* $g(X^m)$*,* $S(X)$ *or* $T(X)$ *is a complete subspace of* X*, then* $\{f, S\}$ *and* $\{g, T\}$ *have a unique multipled point of coincidence in* X.

Moreover, if $\{f, S\}$ *and* $\{g, T\}$ *are w-compatible, then* f*,* g*,* S *and* T *have a unique common multipled fixed point* $(u, \ldots, u) \in X^m$ *and for every* $(x_0^1, x_0^2, \ldots, x_0^m) \in X^m$*, the sequences* $\{x_n\} = \{(x_n^1, x_n^2, \ldots, x_n^m)\} \subset X^m$ *and* $\{u_n\} = \{u_n^1, u_n^2, \ldots, u_n^m\}$ *defined by*

$$\begin{cases} u_{2n-1}^i := ft_i x_{2n-2} = Tx_{2n-1}^i \\ u_{2n}^i := gt_i x_{2n-1} = Sx_{2n}^i \end{cases} \quad \forall i = 1, 2, \ldots, m \tag{11}$$

converge both to $(u, u, \ldots, u)$.

Proof. From (10) and by simple permutations, we have

$$\begin{aligned} d(ft_k x, gt_k u) \le & \sum_{i=1}^m p_i d(St_{ki} x, Tt_{ki} u) + qd(ft_k x, Sx_k) + rd(gt_k u, Tu_k) \\ & + t[d(ft_k x, Tu_k) + d(gt_k u, Sx_k)] \end{aligned}$$

for every $k \in \{1, \ldots, m\}$. Summing the m inequalities,

$$\begin{aligned} \sum_{i=1}^m d(f(t_i x), g(t_i u)) \le & \left(\sum_{i=1}^m p_i\right) \sum_{i=1}^m d(Sx_i, Tu_i) + q \sum_{i=1}^m d(ft_i x, Sx_i) \\ & + r \sum_{i=1}^m d(gt_i u, Tu_i) + t \left[\sum_{i=1}^m d(ft_i x, Tu_i) + \sum_{i=1}^m d(gt_i u, Sx_i)\right]. \end{aligned}$$

In view of (8) and (9),

$$D(\tilde{f}x, \tilde{g}u) \le \left(\sum_{i=1}^m p_i\right) D(\tilde{S}x, \tilde{T}u) + qD(\tilde{f}x, \tilde{S}x) + rD(\tilde{g}u, \tilde{T}u) + t[D(\tilde{f}x, \tilde{T}u) + D(\tilde{g}u, \tilde{S}x)],$$

where $\tilde{f}$, $\tilde{g}$, $\tilde{S}$ and $\tilde{T}$ are defined for all $x = (x_i)_{1 \le i \le m} \in X^m$ and $u = (u_i)_{1 \le i \le m} \in X^m$ by

$$\begin{cases} \tilde{f}(x) = (ft_1 x, ft_2 x, \ldots, ft_m x) \\ \tilde{g}(x) = (gt_1 x, gt_2 x, \ldots, gt_m x) \\ \tilde{S}(u) = (Su_1, Su_2, \ldots, Su_m) \\ \tilde{T}(u) = (Tu_1, Tu_2, \ldots, Tu_m). \end{cases}$$

The contractive condition in Corollary 10 is satisfied for $\tilde{f}$, $\tilde{g}$, $\tilde{S}$ and $\tilde{T}$.

We have $\begin{cases} f(X^m) \subset T(X) \Longrightarrow \tilde{f}(X^m) \subset \tilde{T}(X^m) \\ g(X^m) \subset S(X) \Longrightarrow \tilde{g}(X^m) \subset \tilde{S}(X^m). \end{cases}$

If one of $f(X^m)$, $g(X^m)$, $S(X)$ or $T(X)$ is complete then $\tilde{f}(X^m)$, $\tilde{g}(X^m)$, $\tilde{S}(X^m)$ or $\tilde{T}(X^m)$ is complete in X^m, hence by Corollary 10 applied to the product cone b-metric space X^m, the pairs $\{\tilde{f}, \tilde{S}\}$ and $\{\tilde{g}, \tilde{T}\}$ have unique points of coincidence which are, by Lemma 16, the unique multipled points of coincidence of $\{f, S\}$ and $\{g, T\}$.

If in addition $\{f, S\}$ and $\{g, T\}$ are w-compatible then, from Lemma 16, $\{\tilde{f}, \tilde{S}\}$ and $\{\tilde{g}, \tilde{T}\}$ are w-compatible. By Corollary 10, $\tilde{f}$, $\tilde{g}$, $\tilde{S}$ and $\tilde{T}$ have a unique common fixed point which is the unique multiple common fixed point of f, g, S and T. Because of the uniqueness, from Remark 17, it is of the form $(u, \ldots, u)$, for some $u \in X$.

Also, from Corollary 10, for any $(x_0^1, x_0^2, \ldots, x_0^m) \in X^m$, the sequences $\{x_n\} = \{(x_n^i)_{1 \le i \le m}\}$ and $\{u_n\} = \{(u_n^i)_{1 \le i \le m}\}$ defined by

$$\begin{cases} u_{2n-1} := \tilde{f}x_{2n-2} = \tilde{T}x_{2n-1} \\ u_{2n} := \tilde{g}x_{2n-1} = \tilde{S}x_{2n} \end{cases} \tag{12}$$

converge to $(u, \dots, u) \in X^m$ and (12) is equivalent to

$$\begin{cases} u^i_{2n-1} := f t_i x_{2n-2} = T x^i_{2n-1} \\ u^i_{2n} := g t_i x_{2n-1} = S x^i_{2n} \end{cases} \quad \forall i = 1, 2, \dots, m.$$

Hence the sequences $\{x_n\}$ and $\{u_n\}$ defined in (11) converge to $(u, u, \dots, u) \in X^m$. □

Theorem 21 extends Theorem 3.6 of Olaoluwa and Olaleru [19] to cone b-metric spaces. It also extends and improves Theorem 2.4 of Abbas et al. [22] from maps defined in X^2 to maps defined in X^m, $m \geq 2$. The condition of w-compatibility is also replaced with the weaker w^*-compatibility. Other results generalized are Theorems 2.2, 2.5 and 2.6 of Sabetghadam et al. [23] (results on coupled fixed points of one map) among others.

Example 22. Let $X = [0, \infty)$, $E = C^1_{\mathbb{R}}[0, 1]$, $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0, 1]\}$ and $d : X \times X \to E$ defined by $d(x, y) = |x - y|\varphi$, where $\varphi(t) = e^t + 1$. (X, d) is a cone metric space, i.e. a cone b-metric space with coefficient $s = 1$. Let $\alpha, \beta, \gamma \in \mathbb{R}_+$ be such that $\alpha \leq \beta\gamma < \frac{\beta}{m}$, where $m \geq 2$ is an integer. Consider the mappings $f, g : X^m \to X$ and $S, T : X \to X$ defined by $f(x_1, x_2, \dots, x_m) = g(x_1, x_2, \dots, x_m) = \alpha \sum_{i=1}^m x_i$ and $S(x) = T(x) = \beta x$.

$f(X^m) = g(X^m) = X$ which is complete. Also, $\{f, S\}$ and $\{g, T\}$ are w-compatible. The condition (3.3) of Theorem 3.9 is satisfied for $j = 1$, $q = r = t = 0$ and $p_i = \gamma$, $\forall i \in \{1, \dots, n\}$. That is, for all $x = (x_1, \dots, x_m), u = (u_1, \dots, u_m) \in X^m$,

$$d(fx, gu) \leq \sum_{i=1}^{m} \gamma d(Sx_i, Tu_i).$$

Hence F and g have a unique common multipled fixed point $(x^*, \dots, x^*) \in X^m$. It is easy to notice that $x^* = 0$.

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