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RESEARCH ARTICLE

On The Stability of Continuous Block Backward Differentiation Formula For Solving Stiff Ordinary Differential Equations

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This paper focuses on the stability regions of numerical methods and to demonstrate its suitability for the solution stiff ordinary differential equations. In particular, the purpose of this paper is to generate the stability region for methods of Continuous Block Backward Differentiation Formulae (CBBDF) that simultaneously generate the approximate solution of the stiff ODEs. The practical importance of the methods is established for the stability regions since it cover the whole of the negative half-complex plane.

Keywords: Stability region; Continuous block; backward differentiation formulae; Stiff Ordinary Differential Equation ; collocation and interpolation.

AMS Subject Classification: 65L05, 65L06.

1. Introduction

Many numerical techniques have appeared in the literature for the numerical solution of stiff initial value problems (IVPs) and these techniques depend on many factors including rate of convergence, computational cost, data-storage requirements, accuracy, and stability. Chu and Hamilton [1], Shampine and Watts [2] both suggested that the stability problem appears to be the most serious limitation of block methods. Over the last two decades various types of block methods have been developed for the solution of stiff and non-stiff systems of ordinary differential equations (ODEs), for instance [3–5].

Our aim is to investigate the linear stability properties of the continuous block BDF constructed based on collocation and interpolation. Below we give some basic definition of stability of a multistep method given in Lambert [6].

Definition 1.1 The k-step linear multistep method (LMM) for the solution of the differential equation

$$y' = f(t, y), \quad t \in [t_0, T_n], \quad y(t_0) = y_0 \tag{1}$$

where f satisfies a Lipschitz condition as given in Henrici [7] is conventionally written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_k f_{n+j}. \tag{2}$$

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Where h is the step size, $\alpha_k = 1$, α_j, β_k are unknown constants which must be determined and k is the step number of the particular method employed.

Definition 1.2 The first characteristic polynomial ρ of degree k associated with the general method (2), whose coefficients are α_j and the second characteristic polynomial σ whose coefficients are β_j are defined by

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j, \quad \sigma(z) = \sum_{j=0}^{k} \beta_j z^j \tag{3}$$

where $z \in C$ the complex plane. Thus stability is determined by the location of the roots of the characteristic polynomials.

Definition 1.3 The linear multistep method (2) is said to satisfy the root condition if all of the roots of the first characteristic polynomial have modulus less than or equal to unity, and those of modulus unity are simple. The method (2) is said to be zero-stable if it satisfies the root condition.

Definition 1.4 The linear multistep method (2) is said to be absolutely stable in a region S for a fixed $z = h\lambda$ if for all the roots b_i of the stability polynomial

$$\phi(b, z) = \rho(b) - z\sigma(b) = 0 \tag{4}$$

satisfy $|b_i| < 1$, $i = 1, 2, \ldots, k$.

Definition 1.5 A numerical method is said to be A-stable if its region of absolute stability contains the whole left plane.

2. Stability Theory of Block Numerical Methods for ODEs

In this section, we introduce the basic definition of a block method described by Fatunla [8]. Let Y_n and F_n be vectors defined by

$$Y_\mu = [y_{n+j}]^T, \quad j = 1, 2, \ldots, s \tag{5}$$

$$F_\mu = [f_{n+j}]^T, \quad j = 1, 2, \ldots, s \tag{6}$$

respectively. Then a general k-step, continuous block method is a matrix finite difference equation of the form

$$Y_\mu = \sum_{j=0}^{\ell} \eta_j y_{\mu-j} + h \sum_{j=0}^{\ell} \gamma_j f_{\mu-j}. \tag{7}$$

Where all η_j, γ_j are the right $s \times s$ matrix coefficients and $\mu = 0, 1, 2, \ldots$ represent the block number, $n = \mu s$ the first step number in the μ-th block and s is the proposed block size.

3. Derivation of The Method

The block algorithm proposed in this paper is based on interpolation and collocation, see [9–13] and their references therein. we proceed by seeking an approximate of the exact solution $y(t)$ by assuming a continuous solution $Y(t)$ of the form

$$Y(t) = \sum_{j=0}^{q+r-1} m_j \varphi_j(t). \tag{8}$$

Such that $t \in [t_0, T_n]$, m_j are unknown coefficients and $\varphi_j(t)$ are polynomial basis functions of degree $q+r-1$, where the number of interpolation points q and the collocation point r are respectively chosen to satisfy $q = k$ and $r = 1$. The integer $k \geq 1$ denotes the step number of the method. We thus, construct a k-step block methods with $\varphi_j(t) = t^j_{n+i}$ by imposing the following conditions

$$\sum_{j=0}^{q} m_j t^j_{n+i} = y_{n+i}, \quad i = 0, \dots, q-1 \tag{9}$$

$$\sum_{j=0}^{q} m_j j t^{j-1}_{n+i} = f_{n+i}, \quad i = k \tag{10}$$

where y_{n+j} is the approximation for the exact solution $y(t_{n+j})$, $f_{n+j} = f(t_{n+j}, y_{n+j})$, n is the grid index and $t_{n+j} = t_n + jh$. It should be noted that equation (9) and (10) leads to a system of $q+1$ equations of the form $AM = C$ where

$$A = \begin{pmatrix} t^0_n & t_n & t^2_n & \dots & t^q_n \\ t^0_{n+1} & t_{n+1} & t^2_{n+1} & \dots & t^q_{n+1} \\ t^0_{n+2} & t_{n+2} & t^2_{n+2} & \dots & t^q_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^0_{n+q-1} & t_{n+q-1} & t^2_{n+q-1} & \dots & t^q_{q-1} \\ 0 & 1 & 2t_{n+k} & \dots & qt^{q-1}_{n+k} \end{pmatrix}$$

$$M = (m_0, m_1, m_2, \dots, m_k)^T$$

$$C = (y_n, y_{n+1}, y_{n+2}, \dots, y_{n+k-1}, f_n + k)^T$$

which must be solved to obtain the coefficients m_j. After some algebraic computations our k-step continuous block BDF method is then obtained by substituting these values of m_j into equation (4). We then obtained the expression in the form

$$Y(t) = -\sum_{j=0}^{q-1} \alpha_j(t) y_{n+j} + h\beta_k(t) f_{n+k} \tag{11}$$

where $\alpha_j(t)$ and $\beta_k(t)$ are continuous coefficients. The method (11) is then used to generate the standard BDF method of order k at the desired point $t = t_{n+i}$, $i = 1, 2, \dots, q$.

The additional methods are then obtained by evaluating the first derivative of (11) given by (12) at $q-1$ number of points.

$$Y'(t) = \frac{1}{h}(\sum_{j=0}^{q-1} \alpha'_j(t) y_{n+j} + h\beta'_k(t) f_{n+k}). \tag{12}$$

These additional integrators (12) are combined with the standard BDF (11) and implemented as a self starting block method for any desired step number.

For $k = 2$ taking $q = k$, $\varphi_j(t) = t^j_{n+i}$, $i = 0, 1, 2$ and thus evaluating (11) at $t = t_{n+2}$, and combined with (12) at $t = [t_{n+1}]$ we generate the block method (13).

$$\left.\begin{aligned} f_{n+1} &= \tfrac{1}{3h}[hf_{n+2} - 2y_n + 2y_{n+1}] \\ y_{n+2} &= \tfrac{1}{3}[2hf_{n+2} - y_n + 4y_{n+1}] \end{aligned}\right\} \tag{13}$$

Similarly, specifying $k = 3$, $q = k$, $\varphi_j(t) = t^j_{n+i}$, $i = 0, \ldots 3$ and thus evaluating (11) at $t = t_{n+3}$, together with (12) at $t = [t_{n+1},\, t_{n+2}]$ we have the block method (14)

$$\left.\begin{aligned} f_{n+1} &= \tfrac{1}{11h}[-hf_{n+3} - 4y_n - 4y_{n+1} + 8y_{n+2}] \\ f_{n+2} &= \tfrac{1}{22h}[4hf_{n+3} + 5y_n - 28y_{n+1} - 23y_{n+2}] \\ y_{n+3} &= \tfrac{1}{11}[6hf_{n+3} + 2y_n - 9y_{n+1} + 18y_{n+2}] \end{aligned}\right\} \tag{14}$$

4. Stability Analysis

In what follows, the k-step continuous block BDF can be generally rearranged and rewritten as a matrix finite difference equation of the form

$$A^{(1)}Y_{\omega+1} = A^{(0)}Y_\omega + hB^{(1)}F_\omega \tag{15}$$

where

$$Y_{\omega+1} = (y_{n+1}, \ldots, y_{n+k-1}, y_{n+k})^T,$$

$$Y_\omega = (y_{n-k+1}, \ldots, y_{n-1}, y_n)^T,$$

$$F_\omega = (f_{n+1}, \ldots, f_{n+k})^T,$$

for $\omega = 0, \ldots$ and $n = 0, k, \ldots, N - k$. And the matrices $A^{(1)}$, $A^{(0)}$, $B^{(1)}$ are k by k matrices.

Zero stability: It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (15) tends to the difference system

$$A^{(1)}Y_{\omega+1} - A^{(0)}Y_\omega = 0$$

whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(RA^{(1)} - A^{(0)}) = \frac{U}{V}R^{k-1}(1 - R). \tag{16}$$

Following Fatunla [8], the block block method (15) is zero-stable, since from (16), $\rho(R) = 0$ satisfies $|R_j| \leq 1$, $j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1.

For $k = 2$,

$$A^{(1)} = \begin{pmatrix} \frac{-2}{3} & 0 \\ \frac{-4}{3} & 1 \end{pmatrix} \quad A^{(0)} = \begin{pmatrix} 0 & \frac{-2}{3} \\ 0 & \frac{-1}{3} \end{pmatrix}$$

$$\rho(R) = Det[RA^{(1)} - A^{(0)}] = \frac{2R}{3} - \frac{2R^2}{3}. \tag{17}$$

Similarly for $k = 3$,

$$A^{(1)} = \begin{pmatrix} \frac{4}{11} & -\frac{8}{11} & 0 \\ \frac{28}{22} & -\frac{23}{22} & 0 \\ \frac{9}{11} & -\frac{18}{11} & 1 \end{pmatrix} \quad A^{(0)} = \begin{pmatrix} 0 & 0 & \frac{4}{11} \\ 0 & 0 & \frac{5}{22} \\ 0 & 0 & -\frac{2}{11} \end{pmatrix}$$

$$\rho(R) = Det[RA^{(1)} - A^{(0)}] = -\frac{6R^2}{11} + \frac{6R^3}{11}. \tag{18}$$

The zero stability, for $k = 2$ and $k = 3$ are determine from (17) and (18) by setting $\rho(R) = 0$. Thus we have $R = 0$ and $R = 1$ for (17) and $R = 0$ twice and $R = 1$ for (18). Therefore the continuous block BDF is zero stable. Since one of the roots is $+1$.

4.1 *Linear Stability*

The linear stability properties of the continuous block BDF methods are determined by expressing them in the form (15) and applying them to the test problem

$$y' = \lambda y, \quad \lambda < 0.$$

We have the expression

$$Y_{\omega+1} = D(z)Y_{\omega}, \quad z = \lambda h, \tag{19}$$

where the matrix $D(z)$ is given by

$$D(z) = (A^{(1)} + zB^{(1)})^{-1} * A^{(0)}. \tag{20}$$

The matrix $D(z)$ has eigenvalues $\{d_1, \ldots, d_k\} = \{0, \ldots, d_k\}$, where the dominant eigenvalue d_k is the stability function $L(z) : \mathbb{C} \to \mathbb{C}$ which is a rational function with real coefficients. For $k = 2$, we have that

$$B^{(1)} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & -\frac{2}{3} \end{pmatrix}$$

And from(20) the stability function $R(z)$ is given by

$$L(z) = \frac{2 + z}{2 - 3z + 2z^2}. \tag{21}$$

While for $k = 3$,

$$B^{(1)} = \begin{pmatrix} 1 & 0 & \frac{1}{11} \\ 0 & 1 & -\frac{4}{22} \\ 0 & 0 & -\frac{4}{11} \end{pmatrix}$$

And from (20) the stability function $L(z)$ is given by

$$L(z) = \frac{6 + 6z + 2z^2}{6 - 12z + 11z^2 - 6z^3} \tag{22}$$

in the spirit of Hairer and Wanner [14], the stability regions for both $k = 2$ and $k = 3$ are drawn using the equations (21) and (22) as shown in Fig. 1 and Fig.2 respectively. In Figures below, the rectangles represent the zeros and plus signs represent the poles of (21) and (22). The plots in white on the left half of the complex plane represent the stability region which corresponds to the stability function (21) and (22) respectively. Clearly, from the figures, it is obvious that methods (13) and (14) are A-stable since according to Hairer and Wanner [14] there are no poles of the stability functions in the left half complex plane.

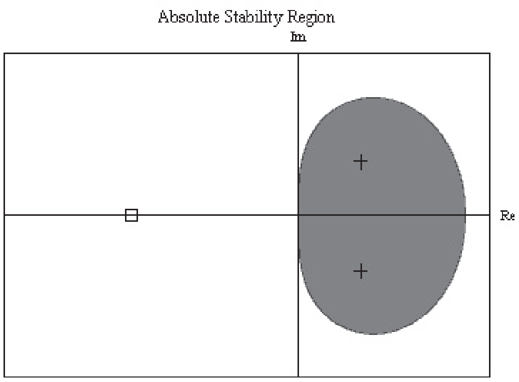

Figure 1.: Stability Region $k = 2$.

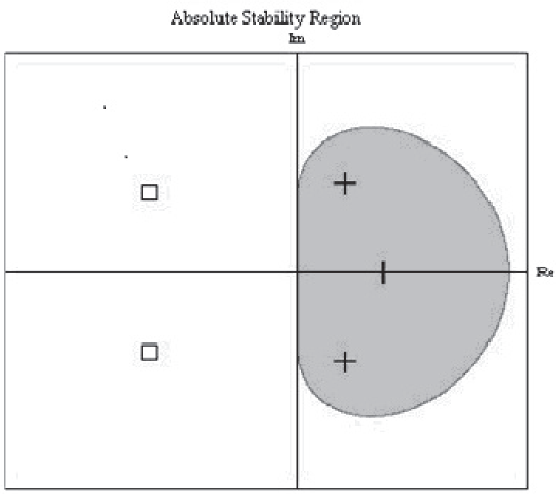

Figure 2.: Stability Region $k = 3$.

5. Numerical Experiments

This section deals with some numerical experiments which illustrate the results derived in the previous sections. Our main aim is to show the good stability properties of the continuous block BDF method.

Experiment 5.1 Experiment 5.1 Consider the systems of first order differential equations on the range $0 \leq t \leq 10$,

$$y_1' = 198y_1 + 199y_2, \quad y_1(0) = 1 \quad y_2' = -398y_1 - 399y_2, \quad y_2(0) = -1.$$

With solution

$$y_1(t) = e^{-t}, \quad y_2(t) = -e^{-t} \text{ and } \lambda = 1, 200.$$

This problem has also been solved by Zarina et al. [5] using implicit r-point block BDF method that was implemented using starting values generated from other method. Their results are here reproduced in Table 2, 3 and For different choices of the constant stepsize h the maximum absolute error is compared with our methods that are implemented as a self starting methods without the use of starting values except the initial value from the problem.

Table 1.: Maximum error and the rate of convergence $ROC = log_2(\frac{e^{2h}}{e^h})$, e^h is the maximum absolute error for h, for Continuous Block BDF of order 2 and 3 for Experiment 5.1

	$k=2$			$k=3$		
h	Step	Maximum Error	Rate	Step	Maximum Error	Rate
0.1	50	6.2×10^{-4}	–	33	4.7×10^{-5}	–
0.05	100	1.5×10^{-4}	2.0	66	5.9×10^{-6}	3.0
0.025	200	3.8×10^{-5}	2.0	133	7.2×10^{-7}	3.0
0.0125	400	9.6×10^{-6}	2.0	266	9.0×10^{-8}	3.0

Table 2.: A comparison of methods for Experiment 5.1, $MaxError_{1t \leq 10} = |y_i - y(t_i)|$

h	Step	Our method Maximum Error $k=2$	Zarina et-al Maximum Error $k=2$
0.01	500	6.13171×10^{-6}	7.18323×10^{-3}
0.001	5000	6.13133×10^{-8}	7.34012×10^{-4}
0.0001	50000	6.14110×10^{-10}	7.35584×10^{-5}

Experiment 5.2 As our second test experiment, we solve the given stiff parabolic equation (see Cash[15]) via the method of lines technique; where we discretize the space derivatives in such a way that the resulting system of ordinary differential equations (ODEs) is stable. We then discretize the time derivatives using the CBBDF which provides multiple discrete methods that are combined and applied as a single matrix equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(1,t), \quad u(x,0) = \sin \pi x + \sin \omega \pi x, \ \omega \gg 1.$$

The exact solution $u(x,t) = e^{-\pi^2 \kappa t} \sin \pi x + e^{-\omega^2 \pi^2 \kappa t} \sin \omega \pi x$.

Table 3.: A comparison of methods for Experiment 5.1, $MaxError_{1t\leq 10} = |y_i - y(t_i)|$

h	Step	Our method Maximum Error $k = 3$	Zarina et-al Maximum Error $k = 3$
0.01	333	4.61670×10^{-8}	1.07308×10^{-2}
0.001	3333	4.60608×10^{-11}	1.10060×10^{-3}
0.0001	33333	6.60305×10^{-13}	1.10333×10^{-4}

Cash[15] notes that as ω increases, equations of the type given in Experiment 5.2 exhibit characteristics similar to model stiff equations. Hence, the methods such as the Crank-Nicolson method which are not A-stable are expected to perform poorly. However, we found that our methods are A-stable, and perform relatively well when applied to this problem. This is due to the fact that our methods are applied in block form on non-overlapping intervals, hence the accumulation of errors is reduced as the integration proceeds on the global interval of interest. In table 4, we display the results for $\kappa = 1$ and a range of values for ω.

Table 4.: A comparison of errors of methods for Experiment 5.2 at $t = 1$ and $\omega = 1$, $\Delta x = 0.1$, $\Delta t = 0.1$

ω	Step 2 BDF	Crank-Nicolson	Cash (2.6a, b)	Cash (2.13a, b, c)
1	1.34×10^{-5}	6.20×10^{-5}	3.7×10^{-5}	1.5×10^{-5}
2	6.63×10^{-6}	3.83×10^{-5}	1.8×10^{-5}	7.4×10^{-6}
3	6.65×10^{-6}	9.30×10^{-3}	1.9×10^{-5}	7.4×10^{-6}
5	6.71×10^{-6}	1.80×10^{-1}	1.8×10^{-5}	7.4×10^{-6}
10	6.71×10^{-6}	6.10×10^{-1}	1.8×10^{-5}	7.4×10^{-6}

6. Conclusion

Two continuous Block Backward Differentiation Formulas (CBBDF) have been presented and implemented as self starting methods for solution of ordinary differential equation. The good stability property of our method makes it attractive for numerical solution of stiff problems. We have demonstrated the efficiency of our method over existing methods as shown the tables above.

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