International Journal of
Mathematical Modelling & Computations
Vol. 05, No. 01, Winter 2015, 15- 28

Convergence Theorems for Asymptotically Pseudocontractive Mappings in the Intermediate Sense for the Modified Noor Iterative Scheme

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Abstract. We study the convergence of the modified Noor iterative scheme for the class of asymptotically pseudocontractive mappings in the intermediate sense which is not necessarily Lipschitzian. Our results improves, extends and unifies the results of [Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings,* Mathematical Analysis and Applications, **158 (2)** (1991) 407-413] and [Qin *et al.Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense,* Fixed Point Theory and Applications, 2010, Article ID 186874, 14 pages, (2010)].

Received: 27 Janurary 2014, Revised: 14 April 2014, Accepted: 21 June 2014.

Keywords: Asymptotically pseudocontractive mapping in the intermediate sense, Weak and strong convergence theorems, The modified Noor iterative scheme.

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1. Introduction

We shall use the following notations all through this paper. We always assume that H is a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. We use the symbols $\rightarrow$ and $\rightharpoonup$ to denote strong convergence and weak convergence respectively. $\omega_w(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$. We assume that C is a nonempty closed convex subset of H and $T : C \rightarrow C$ a mapping. In this paper, we denote the fixed point set of T by $F(T)$.

In the sequel, we give the following definitions of some of the concepts that will feature prominently in this study.

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Definition 1.1. Let $T : C \to C$ be a mapping. T is said to be

(1) *L-Lipschitzian* [4] if there exists an $L > 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \quad \forall x, y \in C, \tag{1.1}$$

(2) *pseudocontractive* [4] if for any $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leqslant \|x - y\|^2, \tag{1.2}$$

and it is well known that condition (1.2) is equivalent to the following:

$$\|x - y\| \leqslant \|x - y + s[(I - Tx) - (I - Ty)]\|, \forall s > 0, x, y \in C, \tag{1.3}$$

(3) *strongly pseudocontractive* [4] if there exists a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that for any $x, y \in C$,

$$\langle Tx - Ty, j(x - y)\rangle \leqslant k\|x - y\|^2, \tag{1.4}$$

(4) λ-*strictly pseudocontractive* [4] in the terminology of Browder and Petryshyn (λ-*strictly pseudocontractive, for short)* if there exists $\lambda > 0$ and $j(x - y) \in J(x - p)$ such that for any $x, y \in C$,

$$\langle Tx - Ty, j(x - y)\rangle \leqslant \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2, \tag{1.5}$$

(5) λ-*demicontractive* [4] if $F(T) \neq \emptyset$ and there exists a constant $\lambda > 0$ and $j(x - p) \in J(x - p)$ such that for any $x \in C$, $p \in F(T)$,

$$\langle Tx - p, j(x - p)\rangle \leqslant \|x - p\|^2 - \lambda\|x - Tx\|^2. \tag{1.6}$$

(6) *nonexpansive* [32] if

$$\|Tx - Ty\| \leqslant \|x - y\|, \forall x, y \in C. \tag{1.7}$$

(7)*asymptotically nonexpansive* [32] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leqslant k_n\|x - y\|, \forall n \geqslant 1, x, y \in C. \tag{1.8}$$

(8) *asymptotically nonexpansive in the intermediate sense* [24] if T is continuous and the following inequality holds:

$$\lim_{n\to\infty} \sup \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leqslant 0. \tag{1.9}$$

Observe that if we define

$$\zeta_n = \max\left\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)\right\}, \tag{1.10}$$

then $\zeta_n \to 0$ as $n \to \infty$. Hence, (1.9) can be reduced to

$$\|T^n x - T^n y\| \leqslant \|x - y\| + \zeta_n, \forall n \geqslant 1, x, y \in C. \tag{1.11}$$

(9) *strictly pseudocontractive* [24] if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\| \leqslant \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in C. \tag{1.12}$$

(10) *asymptotically strict pseudocontraction* [24] if there exist a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leqslant k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \forall n \geqslant 1, x, y \in C. \tag{1.13}$$

(11) *asymptotically strict pseudocontraction in the intermediate sense* [24] if there exist a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \sup \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - k_n\|x - y\|^2 - k\|(I - T^n)x - (I - T^n)y\|^2) \leqslant 0. \tag{1.14}$$

Put

$$\zeta_n = \max\left\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - k_n\|x - y\|^2 - k\|(I - T^n)x - (I - T^n)y\|^2)\right\}. \tag{1.15}$$

It follows that $\zeta_n \to 0$ as $n \to \infty$. Then, (1.14) is reduced to the following:

$$\|T^n x - T^n y\|^2 \leqslant k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 + \zeta_n, \forall n \geqslant 1, x, y \in C. \tag{1.16}$$

(12) *asymptotically pseudocontractive* [24] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, x - y\rangle \leqslant k_n\|x - y\|^2, \quad \forall n \geqslant 1, x, y \in C. \tag{1.17}$$

Observe that (1.17) is equivalent to

$$\|T^n x - T^n y\|^2 \leqslant (2k_n - 1)\|x - y\|^2 + \|x - y - (T^n x - T^n y)\|^2, \forall n \geqslant 1, x, y \in C. \tag{1.18}$$

(13) *asymptotically pseudocontractive mapping in the intermediate sense* [24] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \sup \sup_{x,y\in C} (\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2) \leqslant 0. \tag{1.19}$$

Put

$$\tau_n = \max\left\{0, \sup_{x,y\in C} (\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2)\right\}. \tag{1.20}$$

It follows that $\tau_n \to 0$ as $n \to \infty$. Hence, (1.19) is reduced to the following:

$$\langle T^n x - T^n y, x - y\rangle \leqslant k_n\|x - y\|^2 + \tau_n, \forall n \geqslant 1, x, y \in C. \tag{1.21}$$

In real Hilbert spaces, we observe that (1.21) is equivalent to

$$\|T^n x - T^n y\|^2 \leqslant (2k_n - 1)\|x-y\|^2 + \|(I-T^n)x - (I-T^n)y\|^2 + 2\tau_n, \forall n \geqslant 1, x, y \in C. \tag{1.22}$$

Definition 1.2. [4] A Banach space E is said to satisfy the *Opial condition* if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, the following inequality holds:

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|$$

for any $y \in E$ with $y \neq x$.

Definition 1.3. [28] Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively and let C be a closed convex subset of H. For every $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leqslant \|x - y\| \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C.

Goebel and Kirk [7] introduced the class of asymptotically nonexpansive mappings as a generalization of the class of nonexpansive mappings. They established that if C is a nonempty closed convex and bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then T has a fixed point. The class of asymptotically nonexpansive mapping in the intermediate sense was introduced by Bruck *et al.* [3] in 1993. In 1974, Kirk [11] proved that if C is a nonempty close convex subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. We remark that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings. The class of strict pseudocontractive maps was introduced by Browder and Petryshyn [2]. Marino and Xu [13] established that the fixed point set of strict set of strict pseudocontractions is closed convex, and they obtained a weak convergence theorem for strictly pseudocontractive mappings by Mann iterative process.

The class of asymptotically strict pseudocontractive mappings was introduced by Liu [12]. Sahu *et al.* [28], introduced the class of asymptotically strict pseudocontractive mappings in the intermediate sense in 2009. The class of asymptotically nonexpansive mapping was introduced by Schu [29]. Rhoades [27] produced an example to show that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings. The class of asymptotically pseudocontractive mappings in the intermediate sense was introduced by Qin *et al.* [24]. They obtained some convergence results of Ishikawa iterative processes for the class of mappings which are asymptotically pseudocontractive mappings in the intermediate sense. Olaleru and Okeke [21] introduced the class of asymptotically demicontractive mappings in the intermediate sense and the class of asymptotically hemicontractive mappings in the intermediate sense. We established some interesting fixed points results for this class of nonlinear mappings (see, [21]).

Noor *et al.* [18] gave the following three-step iteration process for solving nonlinear operator equations in real Banach spaces. Let $T : C \to C$ be a mapping. For an arbitrary $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty} \subset C$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, n \geqslant 0, \end{cases} \tag{1.23}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are three sequences satisfying $\alpha_n, \beta_n, \gamma_n \in [0,1]$ for each n.

It was established by Bnouhachem *et al.* [1] that three-step method performs better than two-step and one-step methods for solving variational inequalities. Glowinski and P. Le Tallec [6] applied three-step iterative sequences to finding the approximate solutions of the elastoviscoplasticity problem, eigenvalue problems and in the liquid crystal theory. Moreover, three-step schemes are natural generalization of the splitting methods to solve partial differential equations. What this means is that Noor three-step methods are robust and more efficient than the Mann (one-step) and Ishikawa (two-step) type schemes for solving problems in pure and applied sciences.

The following results will be useful to us in this study.

Lemma 1.4. [24]. Let $\{r_n\}, \{s_n\}$, and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:

$$r_{n+1} \leqslant (1+s_n)r_n + t_n, \forall n \geqslant n_0, \tag{1.24}$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 1.5. [24]. In a real Hilbert space, the following inequality holds:

$$\|ax+(1-a)y\|^2 = a\|x\|^2+(1-a)\|y\|^2-a(1-a)\|x-y\|^2, \forall a \in [0,1], x, y \in C. \tag{1.25}$$

We will always use M to denote $(diam\ C)^2$ henceforth.

Lemma 1.6. [24]. Let C be a nonempty close convex subset of a real Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\}$ and $\{\tau_n\}$ as defined in (1.21). Then $F(T)$ is a closed convex subset of C.

Lemma 1.7. [24]. Let C be a nonempty close convex subset of a real Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense such that $F(T)$ is nonempty. Then $I - T$ is demiclosed at zero.

In 2009, D. R. Sahu *et al.* [28] proved the following theorem on the modified Mann iteration process.

Theorem SXY. Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < \delta \leqslant \alpha_n \leqslant 1-k-\delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C

generated by the modified Mann iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \quad \forall n \in \mathbf{N}. \tag{1.26}$$

Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Qin *et al.* [24] proved the following theorem:

Theorem QCK. Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\tau_n\} \subset [0, \infty)$ defined as in (1.21). Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, n \geqslant 1, \end{cases} \tag{$*$}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following restrictions are satisfied:

(a) $\sum_{n=1}^{\infty} \tau_n < \infty, \sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, where $q_n = 2k_n - 1$ for each $n \geqslant 1$;

(b) $a \leqslant \alpha_n \leqslant \beta_n \leqslant b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$,

then the sequence $\{x_n\}$ generated by $(*)$ converges weakly to fixed point of T.

Consider the following modified Noor iterative scheme:

Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and *asymptotically pseudocontractive mapping in the intermediate sense* with sequences $\{k_n\} \subset [1, \infty)$ and $\tau_n \subset [0, \infty)$ defined as in (1.21) such that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C generated by the following Noor iterative process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geqslant 0, \end{cases} \tag{1.27}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are three sequences satisfying $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ for each n.

In this paper, we study the convergence of the modified Noor iterative scheme (1.27) for the class of asymptotically pseudocontractive mappings in the intermediate sense. Our results improve and extend many others previously announced by other authors.

2. Main Results

Theorem 2.1. Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and *asymptotically pseudocontractive mapping in the intermediate sense* with sequences $\{k_n\} \subset [1, \infty)$ and $\tau_n \subset [0, \infty)$ defined as in (1.21) such that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C generated by the following Noor iterative process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, n \geqslant 0, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are three sequences satisfying $\alpha_n, \beta_n, \gamma_n \in [0,1]$ for each n. Assume that the following restrictions are satisfied:

(i) $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty}(q_n^3 - 1) < \infty$, where $q_n = 2k_n - 1$ for each $n \geqslant 1$;

(ii) $a \leqslant \alpha_n \leqslant \beta_n \leqslant \gamma_n \leqslant b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1+L^2} - 1])$,

then the sequence $\{x_n\}$ generated by (2.1) converges weakly to fixed point of T.

Proof. Fix $p \in F(T)$. From Lemma 1.5, (2.1) and (1.22), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\gamma_n)(x_n - p) + \gamma_n(T^n x_n - p)\|^2\\
&= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|T^n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\
&\leqslant (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\{q_n\|x_n - p\|^2 + \|x_n - T^n x_n\|^2 + 2\tau_n\}\\
&\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\
&\leqslant q_n\|x_n - p\|^2 + \gamma_n\|x_n - T^n x_n\|^2 + 2\gamma_n\tau_n - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\
&\leqslant q_n\|x_n - p\|^2 + \gamma_n^2\|T^n x_n - x_n\|^2 + 2\tau_n. \qquad (2.2)
\end{aligned}$$

$$\begin{aligned}
\|z_n - T^n z_n\|^2 &= \|(1-\gamma_n)(x_n - T^n z_n) + \gamma_n(T^n x_n - T^n z_n)\|^2\\
&= (1-\gamma_n)\|x_n - T^n z_n\|^2 + \gamma_n\|T^n x_n - T^n z_n\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\
&\leqslant (1-\gamma_n)\|x_n - T^n z_n\|^2 + \gamma_n^3 L^2\|x_n - T^n x_n\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2. \qquad (2.3)
\end{aligned}$$

Using Lemma 1.5, (1.22), (2.1), (2.2) and (2.3), we obtain:

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p)\|^2\\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n z_n - p\|^2 - \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2\\
&\leqslant (1-\beta_n)\|x_n - p\|^2 + \beta_n\{q_n\|z_n - p\|^2 + \|z_n - T^n z_n\|^2 + 2\tau_n\}\\
&\quad -\beta_n(1-\beta_n)\|T^n z_n - x_n\|^2\\
&\leqslant (1-\beta_n)\|x_n - p\|^2 + \beta_n\{q_n(q_n\|x_n - p\|^2 + \gamma_n^2\|T^n x_n - x_n\|^2 + 2\tau_n) +\\
&\quad (1-\gamma_n)\|x_n - T^n z_n\|^2 + \gamma_n^3 L^2\|x_n - T^n x_n\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 + 2\tau_n\}\\
&\leqslant q_n^2\|x_n - p\|^2 + \beta_n q_n\gamma_n^2\|T^n x_n - x_n\|^2 + 2q_n\tau_n\\
&\quad +\beta_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + \beta_n\gamma_n^3 L^2\|x_n - T^n x_n\|^2 -\\
&\quad \beta_n\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 + 2\tau_n\\
&\leqslant q_n^2\|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n-\gamma_n q_n - \gamma_n^2 L^2)\|T^n x_n - x_n\|^2\\
&\quad +\beta_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + 2\tau_n(1+q_n). \qquad (2.4)
\end{aligned}$$

Using Lemma 1.5, (1.22), (2.1) and (2.3), we have

$$\begin{aligned}
\|y_n - T^n y_n\|^2 &= \|(1-\beta_n)(x_n - T^n y_n) + \beta_n(T^n z_n - T^n y_n)\|^2\\
&= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|T^n z_n - T^n y_n\|^2\\
&\quad -\beta_n(1-\beta_n)\|T^n z_n - x_n\|^2\\
&\leqslant (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2\|z_n - T^n z_n\|^2\\
&\quad -\beta_n(1-\beta_n)\|T^n z_n - x_n\|^2\\
&\leqslant (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2\{(1-\gamma_n)\|x_n - T^n z_n\|^2 +\\
&\quad \gamma_n^3 L^2\|x_n - T^n x_n\|^2 - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\} -\\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2\\
&= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2(1-\gamma_n)\|x_n - T^n z_n\|^2 -\\
&\quad \beta_n^3 L^2\gamma_n(1-\gamma_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 -\\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2. \qquad (2.5)
\end{aligned}$$

Using (1.22), (2.4) and (2.5), we have

$$\begin{aligned}
\|T^n y_n - p\|^2 &\leqslant q_n\|y_n - p\|^2 + \|y_n - T^n y_n\|^2 + 2\tau_n \\
&\leqslant q_n\{q_n^2\|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad \beta_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + 2\tau_n(1+q_n)\} + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2(1-\gamma_n)\|x_n - T^n z_n\|^2 - \\
&\quad \beta_n^3 L^2\gamma_n(1-\gamma_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 - \\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 + 2\tau_n \\
&= q_n^3\|x_n - p\|^2 - \beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad \beta_n q_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + 2q_n\tau_n(1+q_n) + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2(1-\gamma_n)\|x_n - T^n z_n\|^2 - \\
&\quad \beta_n^3 L^2\gamma_n(1-\gamma_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 - \\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 + 2\tau_n \\
&\leqslant q_n^3\|x_n - p\|^2 - \beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + 2\tau_n(1+q_n+q_n^2). \qquad (2.6)
\end{aligned}$$

Using Lemma 1.5, (1.22) and (2.6), we obtain:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\leqslant (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\{q_n^3\|x_n - p\|^2 - \\
&\quad \beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + 2\tau_n(1+q_n+q_n^2)\} - \\
&\quad \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \qquad (2.7) \\
&\leqslant q_n^3\|x_n - p\|^2 - \alpha_n\beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad \alpha_n(1-\beta_n)\|x_n - T^n y_n\|^2 + 2\tau_n(1+q_n+q_n^2) - \\
&\quad \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\leqslant q_n^3\|x_n - p\|^2 - \alpha_n\beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad 2\tau_n(1+q_n+q_n^2). \qquad (2.8)
\end{aligned}$$

From condition (b), we observe that there exists $n_0 \in \mathbb{N}$

$$\alpha_n\beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2) \geqslant \frac{1-2b-L^2 b^2}{2} > 0, \forall n \geqslant n_0. \qquad (2.9)$$

We note that

$$\|x_{n+1} - p\|^2 \leqslant \left[1 + (q_n^3 - 1)\right]\|x_n - p\|^2 + 2\tau_n(1+q_n+q_n^2), \forall n \geqslant n_0. \qquad (2.10)$$

Using Lemma 1.4, we see that $\lim_{n\to\infty}\|x_n - p\|$ exists. For each $n \geqslant n_0$, we observe that

$$\begin{aligned}
\tfrac{a^2(1-2b-L^2b^2)}{2}\|T^n x_n - x_n\|^2 &\leqslant (q_n^3 - p)\|x_n - p\|^2 + \|x_n - p\|^2 - \\
&\quad \|x_{n+1} - p\|^2 + 2\tau_n(1+q_n+q_n^2), \qquad (2.11)
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|T^n x_n - x_n\|^2 = 0. \qquad (2.12)$$

Note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leqslant \alpha_n\|T^n y_n - x_n\| \leqslant \alpha_n(\|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\|) \\
&\leqslant \alpha_n(L\|y_n - x_n\| + \|T^n x_n - x_n\|) \\
&\leqslant (1+\beta_n L)\|T^n x_n - x_n\|. \qquad (2.13)
\end{aligned}$$

Hence, from (2.12) we obtain

$$\lim_{n\to\infty}\|T^n x_n - x_n\|^2 = 0. \qquad (2.14)$$

By triangle inequality, we have

$$
\begin{aligned}
\|x_n - Tx_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\
&\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\
&\leqslant (1+L)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\
&\quad + L\|T^n x_n - x_n\|. \qquad (2.15)
\end{aligned}
$$

From (2.12) and (2.14), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\|^2 = 0. \qquad (2.16)$$

But $\{x_n\}$ is bounded, hence we observe that there exists a susequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$. From Lemma 1.7, we have that $x^* \in F(T)$.

We now prove that $\{x_n\}$ converges weakly to x^*. Next, we prove that x^* is unique. Suppose that there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $x' \in C$ and $x^* \neq x'$. From Lemma 1.7, we can show that $x' \in F(T)$. Put $d = \lim_{n\to\infty} \|x_n - x^*\|$. Since H satisfies Opial property, we see that

$$
\begin{aligned}
d &= \lim_{n_i\to\infty} \inf \|x_{n_i} - x^*\| < \lim_{n_i\to\infty} \inf \|x_{n_i} - x'\| \\
&= \lim_{n_j\to\infty} \inf \|x_{n_j} - x'\| < \lim_{n_j\to\infty} \inf \|x_{n_j} - x^*\| \\
&= \lim_{n_i\to\infty} \inf \|x_{n_i} - x^*\| = d. \qquad (2.17)
\end{aligned}
$$

Which is a contradiction. It follows that $x^* = x'$. The proof of the theorem is complete.

Next, we establish the hybrid Noor algorithm for L-Lipschitzian asymptotically pseudocontractive mappings in the intermediate sense to obtain a strong convergence theorem without any compact assumption.

Theorem 2.2. Let C be a nonempty closed convex bounded subset of a real Hilbert space H, P_C the metric projection from H onto C, and $T : C \to C$ a uniformly L-Lipschitz and *asymptotically pseudocontractive mapping in the intermediate sense* with sequences $\{k_n\} \subset [1,\infty)$ and $\tau_n \subset [0,\infty)$ defined as in (1.21). Let $q_n = 2k_n - 1$ for each $n \geqslant 1$. Assume that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C generated by the following hybrid Noor algorithm:

$$
\begin{cases}
y_n = (1-\alpha_n)x_n + \alpha_n T^n z_n \\
z_n = (1-\beta_n)x_n + \beta_n T^n s_n \\
s_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, n \geqslant 0, \\
C_n = \{u \in C : \|y_n - u\|^2 \leqslant \|x_n - u\|^2 + \alpha_n\theta_n \\
\qquad + \alpha_n\beta_n\gamma_n q_n(\gamma_n + \gamma_n q_n + \gamma_n^2 L^2 - 1)\|T^n x_n - x_n\|^2\} \\
Q_n = \{u \in C : \langle x_1 - x_n, x_n - u\rangle \geqslant 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_1,
\end{cases}
\qquad (2.18)
$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are three sequences satisfying $\alpha_n, \beta_n, \gamma_n \in [0,1]$ for each n and $\theta_n = q_n([1+\beta_n(q_n-1)]-1)M + 2\tau_n(1+q_n+q_n^2)$ for each $n \geqslant 1$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are chosen such that $a \leqslant \alpha_n \leqslant \beta_n \leqslant \gamma_n \leqslant b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1+L^2}-1])$. Then the sequence generated by (2.18) converges strongly to a fixed point of T.

Proof. The proof is divided into seven steps.

Step 1. We must show that $C_n \cap Q_n$ is closed and convex for all $n \geqslant 1$.
By definition of Q_n, it is clear that it is closed and convex and C_n is closed for each $n \geqslant 1$. We, therefore, only need to show that C_n is convex for each $n \geqslant 1$. Observe that

$$C_n = \{u \in C : \|y_n - u\|^2 \leqslant \|x_n - u\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\gamma_n q_n(\gamma_n + \gamma_n q_n + \gamma_n^2 L^2 - 1)\|T^n x_n - x_n\|^2\} \tag{2.19}$$

is equivalent to

$$C_n' = \{u \in C : 2\langle x_n - y_n, u\rangle \leqslant \|x_n\|^2 - \|y_n\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\gamma_n q_n(\gamma_n + \gamma_n q_n + \gamma_n^2 L^2 - 1)\|T^n x_n - x_n\|^2\} \tag{2.20}$$

Clearly, we see that C_n' is convex for each $n \geqslant 1$. This implies that $C_n \cap Q_n$ is closed and convex for each $n \geqslant 1$. This completes Step 1.
Step 2. We must show that $F(T) \subset C_n \cap Q_n$ for each $n \geqslant 1$.

Let $p \in F(T)$. From Lemma 1.3 and the algorithm (2.18), we observe that

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n z_n - p)\|^2\\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n z_n - p\|^2 - \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2\\ &\leqslant (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(q_n\|z_n - p\|^2 + \|z_n - T^n z_n\|^2 + 2\tau_n) -\\ &\quad \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2\\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n q_n\|z_n - p\|^2 + \alpha_n\|z_n - T^n z_n\|^2 +\\ &\quad 2\alpha_n\tau_n - \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2. \end{aligned} \tag{2.21}$$

$$\begin{aligned}\|s_n - p\|^2 &= \|(1-\gamma_n)(x_n - p) + \gamma_n(T^n x_n - p)\|^2\\ &= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|T^n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\ &\leqslant (1-\gamma_n)\|x_n - p\|^2 + \gamma_n(q_n\|x_n - p\|^2 + \|x_n - T^n x_n\|^2 + 2\tau_n) -\\ &\quad \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\ &= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n q_n\|x_n - p\|^2 + \gamma_n\|x_n - T^n x_n\|^2 +\\ &\quad 2\gamma_n\tau_n - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2. \end{aligned} \tag{2.22}$$

$$\begin{aligned}\|z_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n s_n - p)\|^2\\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n s_n - p\|^2 - \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2\\ &\leqslant (1-\beta_n)\|x_n - p\|^2 + \beta_n(q_n\|s_n - p\|^2 + \|s_n - T^n s_n\|^2 + 2\tau_n) -\\ &\quad \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2\\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n q_n\|s_n - p\|^2 + \beta_n\|s_n - T^n s_n\|^2 +\\ &\quad 2\beta_n\tau_n - \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2. \end{aligned} \tag{2.23}$$

$$\begin{aligned}\|s_n - T^n s_n\|^2 &= \|(1-\gamma_n)(x_n - T^n s_n) + \gamma_n(T^n x_n - T^n s_n)\|^2\\ &= (1-\gamma_n)\|x_n - T^n s_n\|^2 + \gamma_n\|T^n x_n - T^n s_n\|^2 -\\ &\quad \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\\ &\leqslant (1-\gamma_n)\|x_n - T^n s_n\|^2 + \gamma_n L^2\|x_n - s_n\|^2 -\\ &\quad \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2. \end{aligned} \tag{2.24}$$

$$\begin{aligned}\|z_n - T^n z_n\|^2 &= \|(1-\beta_n)(x_n - T^n z_n) + \beta_n(T^n s_n - T^n z_n)\|^2\\ &= (1-\beta_n)\|x_n - T^n z_n\|^2 + \beta_n\|T^n s_n - T^n z_n\|^2 -\\ &\quad \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2\\ &\leqslant (1-\beta_n)\|x_n - T^n z_n\|^2 + \beta_n L^2\|s_n - z_n\|^2 -\\ &\quad \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2. \end{aligned} \tag{2.25}$$

Using (2.22)-(2.25), we obtain:

$$\begin{aligned}
\|y_n - p\|^2 \leqslant{} & (1-\alpha_n)\|x_n - p\|^2 + \alpha_n q_n\|z_n - p\|^2 + \alpha_n\|z_n - T^n z_n\|^2 + \\
& 2\alpha_n\tau_n - \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2 \\
\leqslant{} & (1-\alpha_n)\|x_n - p\|^2 + \alpha_n q_n\{(1-\beta_n)\|x_n - p\|^2 + \beta_n q_n\|s_n - p\|^2 + \\
& \beta_n\|s_n - T^n s_n\|^2 + 2\beta_n\tau_n - \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2\} + \\
& \alpha_n\{(1-\beta_n)\|x_n - T^n z_n\|^2 + \beta_n L^2\|s_n - z_n\|^2 - \\
& \beta_n(1-\beta_n)\|T^n s_n - x_n\|^2\} + 2\alpha_n\tau_n - \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2 \\
={} & (1-\alpha_n)\|x_n - p\|^2 + \alpha_n q_n(1-\beta_n)\|x_n - p\|^2 + \\
& \alpha_n\beta_n q_n^2\|s_n - p\|^2 + \alpha_n\beta_n q_n\|s_n - T^n s_n\|^2 + 2\alpha_n q_n\beta_n\tau_n - \\
& \alpha_n q_n\beta_n(1-\beta_n)\|T^n s_n - x_n\|^2 + \alpha_n(1-\beta_n)\|x_n - T^n z_n\|^2 + \\
& \alpha_n\beta_n L^2\|s_n - z_n\|^2 - \alpha_n\beta_n(1-\beta_n)\|T^n s_n - x_n\|^2 + 2\alpha_n\tau_n - \\
& \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2 \\
\leqslant{} & ((1-\alpha_n) + \alpha_n q_n(1-\beta_n))\|x_n - p\|^2 + \alpha_n\beta_n q_n^2\{(1-\gamma_n)\|x_n - p\|^2 + \\
& \gamma_n q_n\|x_n - p\|^2 + \gamma_n\|x_n - T^n x_n\|^2 + 2\gamma_n\tau_n - \\
& \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\} + \alpha_n\beta_n q_n\{(1-\gamma_n)\|x_n - T^n s_n\|^2 + \\
& \gamma_n L^2\|x_n - s_n\|^2 - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\} + 2\alpha_n q_n\beta_n\tau_n - \\
& \alpha_n q_n\beta_n(1-\beta_n)\|T^n s_n - x_n\|^2 + \alpha_n(1-\beta_n)\|x_n - T^n z_n\|^2 + \\
& \alpha_n\beta_n L^2\|s_n - z_n\|^2 - \alpha_n\beta_n(1-\beta_n)\|T^n s_n - x_n\|^2 + 2\alpha_n\tau_n - \\
& \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2 \\
\leqslant{} & \{(1-\alpha_n) + \alpha_n q_n(1-\beta_n) + \alpha_n\beta_n q_n^2(1-\gamma_n) + \gamma_n\alpha_n q_n^3\}\|x_n - p\|^2 + \\
& \{\alpha_n\beta_n\gamma_n q_n^2 - \alpha_n\beta_n q_n^2\gamma_n(1-\gamma_n) - \alpha_n\beta_n q_n\gamma_n(1-\gamma_n)\}\|T^n x_n - x_n\|^2 \\
& +2\alpha_n\tau_n(1 + \beta_n q_n + \gamma_n\beta_n q_n^2) \\
\leqslant{} & \|x_n - p\|^2 + \alpha_n\beta_n\gamma_n q_n(\gamma_n + \gamma_n q_n + \gamma_n^2 L^2 - 1)\|T^n x_n - x_n\|^2 + \\
& \alpha_n\theta_n. \qquad (2.26)
\end{aligned}$$

where $\theta_n = q_n([1+\beta_n(q_n - 1)] - 1)M + 2(q_n + 1)\tau_n$ for each $n \geqslant 1$. It follows that $p \in C_n$ for all $n \geqslant 1$. This shows that $F(T) \subset C_n$ for all $n \geqslant 1$.

We now show that $F(T) \subset Q_n$ for all $n \geqslant 1$. We prove this by inductions. Clearly, $F(T) \subset Q_1 = C$. Suppose that $F(T) \subset Q_k$ for some $k > 1$. Since x_{k+1} is the projection of x_1 onto $C_k \cap Q_k$, we see that

$$\langle x_1 - x_{k+1}, x_{k+1} - y\rangle \geqslant 0, \quad \forall x \in C_k \cap Q_k.$$

By induction, we know that $F(T) \subset C_k \cap Q_k$. Hence, for each $y \in F(T) \subset C$, we obtain:

$$\langle x_1 - x_{k+1}, x_{k+1} - y\rangle \geqslant 0, \qquad (2.27)$$

this implies that $y \in Q_{k+1}$. Hence, $F(T) \subset C_{k+1}$. This shows that $F(T) \subset Q_n$ for all $n \geqslant 1$. Hence, $F(T) \subset C_n \cap Q_n$ for each $n \geqslant 1$. This completes step 2.

Step 3. We must show that $\lim_{n\to\infty}\|x_n - x_1\|$ exists.

From (2.18), we observe that $x_n = P_{Q_n}x_1$ and $x_{n+1} \in Q_n$ which shos that

$$\|x_1 - x_n\| \leqslant \|x_1 - x_{n+1}\|. \qquad (2.28)$$

Hence the sequence $\|x_n - x_1\|$ is nondecreasing. Recall that C is bounded. This implies that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. This completes step 3.

Step 4. We must show that $x_{n+1} - x_n \to 0$ as $n \to \infty$.

Observe that $x_n = P_{Q_n}x_1$ and $x_{n+1} = P_{C_n\cap Q_n} \in Q_n$. This implies that

$$\langle x_{n+1} - x_n, x_1 - x_n\rangle \leqslant 0, \qquad (2.29)$$

from which it follows that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|(x_{n+1}-x_1)+(x_1-x_n)\|^2\\ &= \|x_{n+1}-x_1\|^2+\|x_1-x_n\|^2+2\langle x_{n+1}-x_1, x_1-x_n\rangle\\ &\leqslant \|x_{n+1}-x_1\|^2-\|x_1-x_n\|^2. \end{aligned}\tag{2.30}$$

Hence, we obtain $x_{n+1}-x_n \to 0$ as $n\to\infty$. This completes step 4.

Step 5. We must show that $T^n x_n - x_n \to 0$ and $T^n s_n - x_n \to 0$ as $n \to \infty$.
In view of $x_{n+1}\in C_n$, we obtain

$$\begin{aligned}\|y_n-x_{n+1}\|^2 \leqslant\ & \|x_n-x_{n+1}\|^2+\alpha_n\theta_n\\ &+[\alpha_n\beta_n^3L^2\gamma_n(q_n\gamma_n+L^2\gamma_n^2-1)]\|T^nx_n-x_n\|^2\\ &+(\alpha_nq_n\beta_n^2-\gamma_n\alpha_nq_n\beta_n^2+\alpha_n\beta_n^3L^2+\alpha_n\beta_n^2-\alpha_n\beta_n)\|T^ns_n-x_n\|^2.\end{aligned}\tag{2.31}$$

On the other hand, we see that

$$\|y_n-x_{n+1}\|^2=\|y_n-x_n+x_n-x_{n+1}\|^2=\|y_n-x_n\|^2+\|x_n-x_{n+1}\|^2+2\langle y_n-x_n, x_n-x_{n+1}\rangle.\tag{2.32}$$

Combining (2.31) and (2.32) and recalling that $y_n=(1-\alpha_n)x_n+\alpha_nT^nz_n$, we obtain:

$$\begin{aligned}\alpha_n\|T^nz_n-x_n\|^2&+2\langle T^nz_n-x_n, x_n-x_{n+1}\rangle\\ &\leqslant \theta_n+[\alpha_n\beta_n^3L^2\gamma_n(q_n\gamma_n+L^2\gamma_n^2-1)]\|T^nx_n-x_n\|^2\\ &\quad+(\alpha_nq_n\beta_n^2-\gamma_n\alpha_nq_n\beta_n^2+\alpha_n\beta_n^3L^2\\ &\quad+\alpha_n\beta_n^2-\alpha_n\beta_n)\|T^ns_n-x_n\|^2.\end{aligned}\tag{2.33}$$

From the asumption, we observe that there exists n_0 such that

$$1-q_n\gamma_n-L^2\gamma_n^2 \geqslant \frac{1-2b-L^2b^2}{2}>0, \forall n\geqslant n_0.\tag{2.34}$$

For any $n \geqslant n_0$, it follows from (2.33) that

$$\frac{a(1-2b-L^2b^2)}{2}\|T^nx_n-x_n\|^2 \leqslant \theta_n+2\|T^nz_n-x_n\|\|x_n-x_{n+1}\|.\tag{2.35}$$

Similarly,

$$\frac{a(1-2b-L^2b^2)}{2}\|T^ns_n-x_n\|^2 \leqslant \theta_n+2\|T^nz_n-x_n\|\|x_n-x_{n+1}\|.\tag{2.36}$$

Note that $\theta_n\to 0$ as $n\to\infty$. Hence, by Step 4, we have:

$$\lim_{n\to\infty}\|T^nx_n-x_n\|=0.\tag{2.37}$$

and

$$\lim_{n\to\infty}\|T^ns_n-x_n\|=0.\tag{2.38}$$

This completes Step 5.

Step 6. We must show that $Tx_n-x_n\to 0$ and $Ts_n-x_n\to 0$ as $n\to\infty$.

$$\begin{aligned}\|x_n-Tx_n\| &\leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|\\ &\quad+\|T^{n+1}x_{n+1}-T^{n+1}x_n\|+\|T^{n+1}x_n-Tx_n\|\\ &\leqslant (1+L)\|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|\\ &\quad+L\|T^nx_n-x_n\|.\end{aligned}\tag{2.39}$$

Similarly,

$$\begin{aligned}\|x_n-Ts_n\| &\leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}s_{n+1}\|\\ &\quad+\|T^{n+1}s_{n+1}-T^{n+1}s_n\|+\|T^{n+1}s_n-Ts_n\|\\ &\leqslant (1+L)\|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}s_{n+1}\|\\ &\quad+L\|T^ns_n-x_n\|.\end{aligned}\tag{2.40}$$

The conclusion follows from Step 5. This completes Step 6.
Step 7. We must show that $x_n \to q$, where $q = P_{F(T)}x_1$ as $n \to \infty$.

From Lemma 1.5, we have that $\omega_w(x_n) \subset F(T)$. From $x_n = P_{Q_n}x_1$ and $F(T) \subset Q_n$, we observe that

$$\|x_1 - x_n\| \leqslant \|x_1 - q\|. \tag{2.41}$$

From Lemma 1.5 of Yanes and Xu [31], we obtain Step 7. The proof of Theorem 2.2 is complete.

Remark 2.3. The results of Theorem 2.2 is more general and it is an improvement of X. Qin *et al.* [24], in the sense that if $\gamma_n = 0\ \forall n \geqslant 1$, then, we obtain the results of Qin *et al.* [24]. If $\tau_n = 0,\ \forall n \geqslant 1$, then we obtain the results of Kim and Xu [10], Marino and Xu [13], Qin *et al.* [24], Sahu *et al.* [28] and Zhou [33].

3. Conclusion

The fixed points results established in this paper improves, generalizes and extends several other fixed point results in literature including Schu [29], Qin *et al.* [24] among others.

Acknowledgements
The authors wish to thank the referees for their comments and observations which lead to the improvement of the manuscript.

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