A NEW COMPOSITE MULTIDERIVATIVE LINEAR MULTISTEP METHODS FOR SOLVING STIFF INITIAL VALUE PROBLEMS.

by

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CERTIFICATION

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A B S T R A C T

Numerical solutions to stiff problems have over the years been a computational problem. Linear Multistep Methods which are popularly used in obtaining solutions to Ordinaty Differential Equations have some drawbacks when applied to stiff Initial Value Problems (IVPs).

A class of Multiderivative Linear Multistep Methods (MLMMs) is developed for integration formulas of orders 2 to 7. The MLMM is applied to stiff problems using Predictor-Corrector formulas. The convergence, consistency and the stability of these methods are investigated and shown to be suitable for stiff IVPs.

The orders two and four of the formulas derived are shown to be most accurate and reliable.

Which are the methods?

Numerical comparison of various methods in terms of accuracy and rate of convergence are considered and the newly developed schemes are found to be more accurate.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 STATEMENT OF PROBLEM

Many physical, biological and management problems giving rise to Ordinary Differential Equations [ODEs] cannot be solved analytically, that is, in closed form.

The focus of interest in this work is the mathematical model

$$\bar{y}' = \bar{f}(x, \bar{y}) \quad , \quad \bar{y}(a) = \bar{\eta} \qquad x \in [a,b] \tag{1.1}$$

where $\bar{y} = y(x_i)$, $i = 1,2,\ldots,k$ are known functions of x, a and η are real known constants given by the vectors

$$\bar{y}(a) = [y_1(a),\ y_2(a), \ldots,\ y_k(a)]$$

$$\bar{\eta} = [\eta_1, \eta_2, \ldots, \eta_k]$$

There are also some ODEs that do have analytical solutions, but with their numerical results being of greater interest and importance. Such problems therefore require the use of numerical methods for their solutions.

Various numerical approaches for dealing with these situations do exist. One class among such schemes is the discrete variable methods [9,12,14,15 and 18]. This class falls into two distinct categories:

a) The One-step or Runge-Kutta (R-K) methods which are essentially substitution methods.

b) The Linear Multistep Methods (LMMs) which are basically polynomial interpolation schemes.

The R-K method has an advantage of being self-starting. However, higher order R-K methods are usually unsuitable for problems with large system of equations because very large problems often have large Lipschitz constants. For such problems, other methods are usually more efficient and would invariably be used instead.

Numerical investigation of ODEs of the type (1.1) can broadly be carried out for problems regarded as stiff and non-stiff [12]. For stiff problems, the class of Multistep Methods has been a class of methods which often is preferred to the R-K method due to the stability regions required by such problems.

Often most Multistep Methods are of Adam's type which are broadly divided into Adam-Moulton and Adam-Basforth Methods [18]. In stiff problems, it is observed that stiffness is a property of mathematical problems and not of the numerical method. Generally stiff equations are problems for which a typical solution is a rapidly decaying exponential [19].

Their investigation numerically are very tedious for this reason. Thus unlike non-stiff equations for which various numerical methods of solutions are known, the stiff problems have received less attention due to the inherent and other difficulties ususally encountered in the procurement of their solutions.

Hence the purpose of this study is to develop a very efficient numerical method for the solutions of stiff problems, an area which is still open for research [9,13].

The approach employed in obtaining the solution of (1.1) is to derive a new version of multistep methods. The usual Linear Multistep Method (LMM) is extended to the case of Multiderivative Linear Multistep Method (MLMM). The MLMM is a scheme recently receiving attention for the solution of stiff IVP. It will be shown that the new set of formulas proposed gives a more accurate result than other known methods.

The work done in the derivation of the formulas proposed usually involve solving linear systems to determine the parameters and constants governing the formula. The variation allowed in the formulas especially the introduction of free parameters help in the accuracy of the method.

Like some other methods, the system (1.1) possesses a Jacobian matrix $J(x)$ involving partial derivative $\partial f/\partial y$ evaluated at $(x,y(x))$. The case study in this work followed from the work done by Cash [2]. He fitted an exponential function into a test problem

$$y' = \lambda y, \qquad y(x_0) = y_0 \tag{1.2}$$

and a general second derivative linear k-step method was used for the case $k = 1$. However, since the method is based on a second derivative formula, the case $k = 2$ is of a paramount interest.

The study also propose a possible extension of the formula with modifications to fitting it to functions other than exponential function.

Thus, appropriate modifications which improve on the efficiency of the method for stiff problems are then carried out having ensured that necessary convergence, consistency and stability conditions are satisfied. The more efficient and reliable algorithm is then coded and implemented on the digital computer.

The idea of integrating trapezoidal rule into the newly developoed scheme for stiff system will be considered in chapter three.. Gear [12,13] stated that trapezoidal rule applied to stiff systems; gives slowly damped oscillatory errors. However, errors are quickly damped for backward Euler scheme. The magnitude of the error in the trapezoidal rule is controlled by step length choice and the result will still be acceptable.

1.2 NUMERICAL STABILITY FOR STIFF EQUATIONS

The basic problem usually encountered when attempting to obtain numerical approximation to the solution y(x) of stiff equation of the form (1.1) is that of numerical stability. The step length used in a numerical method to obtain a solution usually contributes to the stability limitation of a method.

Hence to overcome this stability limitation on the step size, numerical methods for solution of stiff problems have been sought to possess Regions of Absolute Stability (RAS) in the open left half plane $Re(\lambda h) < 0$.

However, in areas of physical importance, that gave rise to system of ODEs, it is often noticed that such system of equations are inherently very stable, but often numerical methods which are been used are impracticable because of severe step-size restriction imposed by the requirement of numerical stability.

This sort of difficulty was first encountered in equations governing masses controlled by spings of different stiffness and it later became known as stiff equation [17].

Consider a test problem

$$\underline{y}' = \underline{A}\,\underline{y} \quad , \qquad \underline{y}(x_0) = \underline{y}_0 \tag{1.3}$$

where $\underline{y} = (y_1, y_2, \ldots, y_n)$, $y_j \varepsilon R$, $\underline{A}$ is n x n real matrix having eigenvalues λ_j , $j = 1, \ldots, n$ contained in the open left half plane.

Assume that

$$\max_{i,j} \left| \frac{Re\ (\lambda_i)}{Re\ (\lambda_j)} \right| \quad \text{is large, then the system}$$

of equation (1.3) is inherently stable and the exact solution

$$y(x) = e^{(x-x_o)A}\ y_o \tag{1.4}$$

tends to zero exponentially as x increases.

The difficulty faced with stiff equations is that though the component of the true solution corresponding to this eigenvalue soon becomes negligible, the restriction on step size imposed by the numerical stability of standard methods requires that $|h\lambda_j|$ remain small in the range of integration.

Dalquist [3], Gear [10, 12] and some others [16, 22] gave various concepts of stability regions which helped in obtaining methods satisfying some sort of stability criteria in the solutions of stiff problems.

In consideration of stiff problems, accuracy is an important factor of solution obtained. Hence, one of the stability property given by Gear suggested that systems are solved by methods whose stability region extends in the negative real direction. Hence solutions to system of stiff problems shall be sought in the left half plane. In order to satisfy good stability criterion and have solution with a very high accuracy, the work in this thesis attempted to introduce some new algorithm based on some of the most recent numerical methods suggested by Cash [2] and others [9, 14, 16].

1.3 LINEAR MULTISTEP METHODS

The Linear Multistep Method (LMM) of step number k or a linear k-step method for solving (1.1) is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.5)$$

where α_j and β_j are constants and not both α_0 and β_0 are zero, that is, $\alpha_0^2 + \beta_0^2 > 0$

In order to remove the arbitrariness of a constant multiplier on (1.5), we take $\alpha_k = +1$. Equation (1.5) is said to be explicit LMM if $\beta_k = 0$ and implicit if otherwise.

Implicit methods require greater computational efforts but are usually more accurate than the explicit oness for a given step number k [18]. Most LMM are formulated by one of the following approaches:

(a) methods of undetermined coefficients,

(b) numerical differentiation, and

(c) numerical integration.

The last of these is the most efficient. It provides the avenues for error estimation and error analysis which in effect facilitates step-size adjustment [9].

The LMM is essentially seen as a polynomial interpolation procedure, whereby the solution or its derivative is replaced by a polynomial of appropriate degree of the independent variable whose derivative or integral is readily obtained.

Clearly, linear k-step methods, k=1,2,... can be derived from various known methods. Such methods of derivation include the Taylor series approach, Newton-Cotes integration formulas and Newton-Gregory interpolation algorithms [15, 21].

The derivation of other LMMs which may not be included by the approaches described above can be found through the determination of the order of such methods [18].

1.4 CLASSIFICATION OF LMM

The LMM known to be vast in procuring solution to IVP can be classified broadly into explicit and implicit schemes. Before the advent of computer, it was a common practice to express the right hand side of a LMM (1.5) in terms of a power series involvoing finite difference operators. However, since digital computer is now available, it is more convenient to compute with a fixed LMM and alter the step length, whenever there is demand for greater accuracy.

The existence of characteristic polynomial in LMM has resulted in further classification of LMM of different step numbers which share a common form of the first characteristic polynomial $\rho(\xi)$ Thus, methods for which all the spurious roots of the first characteristic polynomial which are located at the origin such that

$$\rho(\xi) = \xi^{k} - \xi^{k-1}$$

are called Adams methods. They are known to be zero stable [18]. Adams methods which are explicit are called Adams-Bashforth methods, while the implicit Adams methods are Adams-Moulton.

Comparison of explicit and implicit LMM reveals that the implicit schemes are more accurate than the explicit ones for a given step number k. Furthemore, the highest attainable order for a zero stable method is less in the case of an explicit method than an implicit one. The explicit methods are self starting, hence, are used for the initial evaluation of values especially when implicit scheme is being used. Thus for higher accuracy, the combination of these two methods is often preferred to serve as predictor and corrector schemes. This combination is referred to as Predictor-Corrector (P-C) methods. The explicit method usually serves as the predictor while the corrector method is implicit in nature.

For every implementation of P-C methods, three steps are involved. These are:

Step 1 Predict the starting value $y_{n+k}^{[o]}$ by

$$P: \quad y_{n+k}^{[o]} = h \sum_{j=0}^{k-1} \beta_j f_{n+j} - \sum_{j=0}^{k-1} \alpha_j y_{n+j}$$

Step 2 Evaluation of $f_{n+k}^{[r]}$ by

$$E: \quad f_{n+k}^{[r]} = f(x_{n+k}, y_{n+k}^{[r]}), \quad r = 0,1,2,\ldots$$

Step 3 To correct $y_{n+k}^{[r]}$ by

$$C: \quad y_{n+k}^{[r+1]} = h\beta_k f_{n+k} + \sum_{j=0}^{k-1} (h \beta_j f_{n+j} - \alpha_j y_{n+j})$$

A combination of the three steps is called PEC mode. There are two ways in which this mode can be implemented. This can either be in the mode $P(EC)^n$ or $P(EC)^nE$. The latter is assumed to posses better stability characteristics than the former [18].

Generally, it is observed that LMM though vast is limited in procuring solution for various classes of IVP, especially stiff problems. The draw back centered on the inability of the LMM to cater for the fast decaying exponential property of stiff problems as discussed in section 1.1. Thus due to this limitation of the LMM, the concept of Multiderivative Linear Multistep Method (MLMM) is laterly being considered for procuring solutions of stiff problems. Appropriate modifications that improved on the efficiency of this new method for stiff problems are then carried out.

Having ensured that necessary convergence, consistency and stability conditions are satisfied, the more efficient method is to be coded and implemented on the digital computer.

1.5 MULTIDERIVATIVE FORMULAE

The general Multiderivative Linear Multistep Method (MLMM) is given by the formula

$$\sum_{j=0}^{k} \alpha_j \; y_{n+j} = \sum_{i=0}^{n} h^i \sum_{j=0}^{k} \tau_{i,j} \; f_{n+j}^{(i)} \tag{1.6}$$

where $f_{n+j}^{(i)}$ are the i-th derivative of f(x,y) evaluated at (x_{n+j}, y_{n+j}).

The method of interest in this work, in particular, is the second derivative LMM obtained from (1.6) when n = 2.

Then we have

$$\sum_{j=0}^{k} \alpha_j \; y_{n+j} = h \sum_{j=0}^{k} \beta_j \; f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j \; g_{n+j} \tag{1.7}$$

where $g_{n+j} = g(x,y) \; \Big|_{\,n+j} = \dfrac{df(x,y)}{dx}$

and $\beta_j = \tau_{1,j}$, $\gamma_j = \tau_{2,j}$

$\alpha_k = +1$, and not all γ_j , $j = 0,1,\ldots,k$ are zero satisfying

$\sum_{j=0}^{k} \alpha^2 > 0$

The attainable order for n^{th}-derivative linear k-step method is given by kn+k+n-1 for implicit methods and kn+k-1 for explicit ones. Attempt was made by Cash [2] to fit exponential functions to the composite form of (1.7) with k = 1 for the test problem

$$y' = \lambda y \;, \quad y(0) = 1 \;, \quad q = \lambda h \tag{1.8}$$

Cash's work showed that the method is A-stable for orders 2 and 3 and may be A-stable for order 4.

However, we shall consider case k=2 of (1.7) so as to have higher orders leading to greater accuracy. This shall require greater computational work than the one step case. However it will exhibit a much better accuracy comparable to the analytical results. It will be observed that MLMM exhibits some better accuracy than the usual LMM. This is partly due to higher powers of h introduced in (1.6) for the former method.

Due to the nature of the MLMMs in (1.7), they require greater numerical and analytical work than the LMMs. However, the derivation of these methods reduces the amount of work required on the digital computer and still retain the stability properties for the case $q \to 0$ and $q \to -\infty$ when tested on the scalar problem (1.8). The set of formula to be considered is a pair in the Predictor-Corrector form based on the second derivative LMM of (1.7).

The characteristic polynomials of any of the form (1.7) are given by

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$$

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$$

$$\theta(\xi) = \sum_{j=0}^{k} \gamma_j \xi^j$$

The stability region of the MLMM can be determined by observing that on the boundary R, one of the roots of the polynomial

$$\rho(\xi) - q\sigma(\xi) - q^2\theta(\xi) = 0 \qquad (1.9)$$

will have modulus one.

Hence when MLMM is fitted to scalar problem (1.8) with $q = \lambda h$, then the Region of Absolute Stability (RAS) in the q plane of the MLMM is that region for which the roots of the characteristic polynomial

$$\sum_{j=0}^{k} (\alpha_j - \beta_j q - \gamma_j q^2)$$

lie in the open unit circle.

CHAPTER TWO

TWO-STEP INTEGRATION FORMULAS

2.1 INTRODUCTION

The Linear Multistep Method (LMM) given by equation (1.5) performs poorly when applied to stiff Initial Value Problems (IVPs). Hence a form of the LMM which is an Implicit backward differentiation formula of (1.5), that is

$$\sum_{j=0}^{k} \alpha_j y_{n+j} - h \beta_k f_{n+k} = 0 \qquad \alpha_k = 1 \tag{2.1}$$

was proved to satisfy the definition of stiff stability by Gear [10]. This was however seen to be less efficient for exponentially fitted problems. The stiffly stable LMM requires that the method be implicit. This has led to several investigation into possible ways of stabilising explicit methods by using Jacobian matrix. One means by which this is done is the application of Jacobian matrix on generalizing LMM by calculating the second derivative y''. This led to the use of second derivative multistep formulae.

This kind of methods is implemented using free parameters to allow for exponential fitting. Enright [7] developed a class of stiffly stable k-step second derivative methods of order k+2 and tested with proving results.

2.2 EXPONENTIALLY FITTED FORMULA

The aim in this thesis is to derive exponentially fitted formulae of various orders for which stability requirements are investgated for all choices of the fitting parameters.

The idea of using exponentially fitted formulae for stiff problem in the form (2.1) was first proposed by Liniger and Willoughby [20]. Integration formulae containing free parameters were derived and these parameters were chosen so that a given function exp(q), where q is real, satisfies the integration formula exactly. This was tested on LMM for $k = 1$, however Jackson and Kenue [17] have derived a fourth order exponentially fitted formulae based on a linear 2-step formula and were efficiently A-stable. These methods which have been cited are deficient in the sense that they do not attempt to estimate the local truncation error of the formula being used. This makes control of the step length h being used difficult to choose and can lead to gross inefficiencies if an appropriate value of h is not used. The new methods being descibed in this thesis contain a 'built-in' local error estimate. This error estimate may be used as a basis of a step control procedure, and the resulting variable step formulae can be found to be particularly efficient in the trancient phase of the region of integration.

Based on this idea, Cash [2], in his own work, attempted using MLMM with $k = 1$ in the second derivative formulae. We shall however derive a more difficult exponentially fitted formulas for the case $k = 2$ of the Multiderivative Linear Multistep Methods (MLMMs) using a composite formula.

The class of methods to be derived, using the Composite MLMMs require the maximum attainable order for the case $k = 2$,

$n = 2$ to be seven. Hence various formulas of orders ranging from 2 to 7 will be given.

The methods being proposed here is basically for stiff problems for which exponential fitting is appropriate. These are often recognised through some knowledge of the physical behaviour of the solution. This procedure is preferred because it is usually found that exponentially fitted integration formulae are substantially more efficient than conventional ones. A numerical investigation of the proposed methods will lead to some conclusion on the stability propoerties of the methods.

The integration formulas which shall be used for the exponential fitting shall be as follows:

The predictor method is chosen to coincide with the second derivatve LMM (1.7), given by

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h \sum_{j=0}^{k} \beta_j \, f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j \, g_{n+j} \tag{2.2}$$

while the corrector formula which is a composite MLMM is given by

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h \sum_{j=0}^{k+1} \beta_j \, f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j \, g_{n+j} \tag{2.3}$$

where $g_{n+j} = \left.\frac{df(x,y)}{dx}\right|_{x}^{y}$ and $\beta_{k+1} \neq 0$

so as to differentiate it from the predictor method (2.2).

For the purpose of deriving effective schemes of different orders, the following definition is given.

Define the L operator as

$$L[y(x);h] = \sum_{j=0}^{k} \{ \alpha_j \, y(x+jh) - h \beta_j \, f[x+jh, y(x+jh)] - h^2 \gamma_j \, g[x+jh, y(x+jh)] \}$$

and by using Taylor series expansion and collecting like terms, we can write L as

$$L = c_0 y(x) + hc_1 y'(x) + \ldots + h^r c_r y^{(r)}(x)$$

where

$$c_o = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_\kappa$$

$$c_1 = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k - (\beta_0 + \beta_1 + \cdots + \beta_\kappa)$$

$$c_2 = \tfrac{1}{2}(\alpha_1 + 2^2\alpha_2 + \ldots + k^2\alpha_k) - (\beta_1 + 2\beta_2 + \ldots + k\beta_k)$$

$$c_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \ldots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots$$

$$k^{q-1}\beta_k) - \frac{1}{(q-2)!}(\gamma_1 + 2^{q-2}\gamma_2 + \ldots + k^{q-2}\gamma_k)$$

$$q = 3,4,\ldots,r$$

and the method (2.2) or (2.3) is of order p if

$$c_0 = c_1 = \ldots = c_p = 0 \quad \text{and} \quad c_{p+1} \neq 0$$

Thus to derive a scheme of order p, we set $c_j = 0$, $j = 0,1,\ldots,p$ giving rise to sets of (p+1) equations from which constant parameters α_j, β_j and γ_j are determined according to equation (2.2) or (2.3).

The algorithm required for the study of the stability criterion connects the Predictor and the Corrector formulas together and is given as follows:

A1 Compute $\bar{y}_{n+k}$ as solution of (2.2)

A2 Compute $\bar{f}_{n+k} = f(x_{n+k}, y_{n+k})$ and $\bar{g}_{n+k}$

A3 Compute $\bar{y}_{n+k+1}$ as the solution of

$$\sum_{j=0}^{k-1}[\alpha_j y_{n+j+1} - h\beta_j f_{n+j+1} - h^2\gamma_j g_{n+j+1}]$$

$$= \alpha_k y_{n+k+1} + h\beta_k f_{n+k+1} + h^2\gamma_k g_{n+k+1}$$

A4 Compute $f_{n+k+1} = f(x_{n+k+1}, y(x_{n+k+1}))$

A5 Compute y_{n+k} as the solution of

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j} + h \beta_{k+1} f_{n+k+1}$$

This algorithm shall henceforth be regarded as Composite Integration Formula (CIF) and the complete scheme defined will be referred to as steps A1 - A5.

2.3 DERIVATION OF ORDER 2 SCHEME

The least scheme to be considered is of order 2. The procedure employed in this thesis requires that the Predictor formula be of order (k-1) while the Corrector scheme is of order k.

The predictor formula corresponding to the case k=2, with one free parameter is obtained from equation (2.2) as

$$\alpha_0 y_n + \alpha_1 y_{n+1} + y_{n+2} = h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}) + h^2(\gamma_0 g_n + \gamma_1 g_{n+1} + \gamma_2 g_{n+2}) \qquad (2.5)$$

By localising assumption of having strictly 2-step method, set coefficients of y, f and g at x_{n+1} point to zero, that is

$\alpha_1 = \beta_1 = \gamma_1 = 0$

Also let $\gamma_0 = \gamma_2 = 0$, then we have α_0, α_2, β_0 and β_2 to determine.

But $\alpha_k = \alpha_2 = +1$ and let $\beta_2 = a$ (free parameter).

For order two predictor formula set $c_0 = c_1 = 0$; $c_2 \neq 0$

using equation (2.4) we obtain $\alpha_0 = -1$, $\beta_0 = 2-a$

Thus (2.5) becomes

$$y_{n+2} - y_n = h\,[a f_{n+2} + (2-a) f_n]$$

but $y' = f(x,y)$

that is $y_{n+2} - y_n = h\,[a y'_{n+2} + (2-a) y'_n]$

By exponential fitting, we use equation (1.8) to get step A1 of the Algorithm as

$$\bar{y}_{n+2} - y_n = h\,[a\lambda\,\bar{y}_{n+2} + (2-a)\lambda\, y_n\,]$$

leading to

$$(1-aq)\,\bar{y}_{n+2} = [1 + (2-a)q\,]\,y_n$$

or $$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q}{1 - aq} \qquad (2.6)$$

where $q = \lambda h$

From where step A2 of the CIF is also given.

The solution of the scalar problem (1.8), that is

$$y' = \lambda y \quad \text{is} \quad y = e^{\lambda x}$$

$$\frac{y_{n+1}}{y_n} = \frac{y(x_n+h)}{y(x_n)} = \frac{e^{\lambda(x_n+h)}}{e^{\lambda x}} = e^{\lambda h} = e^q$$

similarly, $$\frac{y_{n+2}}{y_n} = e^{2q} \qquad (2.7)$$

using (2.7) in (2.6), we obtain

$$e^{2q} = \frac{1 + (2-a)q}{1 - aq}$$

solving for parameter a, we get

$$a = \frac{(1 - e^{2q})/q + 2}{1 - e^{2q}} \qquad (2.8)$$

From equation (2.3), the corrector formula corresponding for k=2 is

$$\alpha_0\, y_n + \alpha_1\, y_{n+1} + \alpha_2\, y_{n+2} = h\,(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3})$$
$$+ h^2\,(\gamma_0 g_n + \gamma_1\, g_{n+1} + \gamma_2\, g_{n+2})$$

Again, setting $\alpha_1 = \beta_1 = 0$, $\gamma_0 = \gamma_1 = \gamma_2 = 0$, $\alpha_2 = +1$ and $\beta_3 = r$ (free parameter), then a method of order 2 involving corrector formula requires

$$c_0 = c_1 = c_2 = 0 \qquad \text{and} \qquad c_3 \neq 0$$

This leads to the simultaneous equations

$$\alpha_0 + 1 = 0$$

$$2 - \beta_0 - \beta_2 - r = 0$$

$$2 - 2\beta_2 - 3r = 0$$

solving we have

$$\alpha_0 = 1, \quad \beta_0 = \tfrac{1}{2}(2+r), \quad \beta_2 = \tfrac{1}{2}(2-3r)$$

hence the method becomes

$$y_{n+2} - y_n = h\,[\, r y'_{n+3} + \tfrac{1}{2}(2-3r) y'_{n+2} + \tfrac{1}{2}(2+r) y'_m \,] \tag{2.9}$$

solving for r to get

$$r = \frac{(e^{2q} - 1)/p - e^{2q} - 1}{e^{2q} - \frac{3}{2} e^{2q} + \frac{1}{2}} \tag{2.10}$$

Proceeding on step A3 of the CIF we should have

$$\alpha_0 y_{n+1} + \alpha_1 y_{n+2} + \alpha_2 y_{n+3} - h\,(\beta_0 f_{n+1} + \beta_1 f_{n+2} + \beta_2 f_{n+3})$$
$$- h^2 (\gamma_0 g_{n+1} + \gamma_1 g_{n+2} + \gamma_2 g_{n+3}) = 0$$

This equation is a transpose of equation (2.5) by a step, hence if we solve for the constants α_j, β_j and γ_j with the same condition impose on (2.5), we shall obtain

$$y_{n+3} - y_{n+1} - h(2-a) f_{n+1} - ahf_{n+3} = 0$$

Applying exponential fitting condition to get

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \frac{1 + (2-a)q}{1 - aq} \tag{2.11}$$

This result is identical to the predictor formula (2.6).

Hence following this trend, we observe that

$$\frac{\bar{y}_{n+j+2}}{y_{n+j}} = \frac{1 + (2-a)q}{1 - aq}, \qquad j = 0,1,\ldots,r$$

satisfying the consistency conditions.

There is the need to obtain ratio $\frac{\bar{y}_{n+3}}{y_n}$ in order to compute step A4 of the CIF. Thus from (2.11), using (2.7) we have

$$\frac{\bar{y}_{n+3}}{y_n} = \frac{1 + (2-a)q}{1 - aq} \cdot \frac{y_{n+1}}{y_n} = e^{3q}$$

that is

$$\frac{\bar{y}_{n+3}}{y_n} = \left(\frac{1 + (2-a)q}{1 - aq} \right)^{\frac{3}{2}} = R^*(q) \qquad (2.12)$$

Finally for step A5, equation (2.9) becomes

$$\frac{y_{n+2}}{y_n} \qquad \frac{rq\ (\bar{y}_{n+3}/y_n) + 1 + \frac{1}{2}q(2+r)}{1 - \frac{1}{2}q(2-3r)}$$

substituting (2.12) to get

$$\frac{y_{n+2}}{y_n} = \frac{rq\ R^*(q) + 1 + \frac{1}{2}q(2+r)}{1 - \frac{1}{2}q(2-3r)} = R(q) \qquad (2.13)$$

This unites both the predictor and the corrector formulas. The formula (2.13) derived is capable of procuring solution to stiff problems for which exponential fitting is applicable.

A good numerical method must satisfy some stability properties. It is therefore necessary to find the range of values of a and r with certain limits of q.

To determine the conditions a and r will satisfy, we examine the inequality

$$\left| \frac{y_{n+2}}{y_n} \right| < 1$$

for all q with $Re(q) < 0$

The necessary and sufficient conditions for this inequality to hold are given by Maximum Modulus Theorem.

THEOREM 2.1 (Maximum Modulus Theorem)

Let f be analytic and not constant in a domain D. Then $|f|$ cannot have a local maximum in D.

The proof is given in Beardon text [1].

By Theorem 2.1, these conditions imply that

(i) $R(q) < 1$ on $\mathrm{Re}(q) = 0$

(ii) $R(q)$ is analytic in $\mathrm{Re}(q) < 0$

If (i) holds, it follows that $R(q)$ is analytic as $q \to -\infty$ and thus (i) and (ii) will guarantee A-stability by Theorem 2.1.

Now, for $|R(q)| < 1$, we consider

$$\frac{y_{n+2}}{y_n} - 1 < 0$$

hence from (2.13) we have

$$\frac{\left(\dfrac{1/q + (2-a)}{1/q - a}\right)^{\frac{3}{2}} + \dfrac{1}{q} + \tfrac{1}{2}(2+r) - \dfrac{1}{q} + \tfrac{1}{2}(2-3r)}{1/q - \tfrac{1}{2}(2-3r)} < 0$$

and as $q \to -\infty$, we obtain

$$\frac{r(1-\frac{2}{a})^{\frac{3}{2}} + 2 - r}{\frac{3}{2}r - 1} < 0$$

leading to $r < \frac{2}{3}$ and $a > 0$.

These inequalities may help to determine stability condition. However, by taking limits of a and r as $q \to 0$ and $q \to -\infty$, we obtain from (2.8) and (2.10) that

$$\lim_{q \to 0} a = \lim_{q \to 0} \frac{1 - e^{2q} + 2q}{q(1 - e^{2q})} = 1$$

Also, $$\lim_{q \to -\infty} a = \lim_{q \to -\infty} \frac{(1 - e^{2q})/q + 2}{1 - e^{2q}}$$

Similarly, $$\lim_{q \to 0} r = -\frac{4}{9} \quad , \quad \text{while} \quad \lim_{q \to -\infty} r = -2$$

Taking values of $q \varepsilon (-\infty, 0)$ for a large sample S, we observed that for various values of q, the values of a and r are within the range above, that is $a \varepsilon (1,2)$ and $r \varepsilon (-2, -\frac{4}{9})$

Thus for such sample we have table 2.1 below.

TABLE 2.1
Values of parameters a and r for order 2 scheme

q	a	r
-1	1.3130	-0.7850
-2	1.5373	-1.1105
-5	1.8001	-1.6003
-50	1.9800	-1.9600
-100	1.9900	-1.9800
-200	1.9980	-1.9960

This set of values suggests that our integration formula may be A-stable because according to theorem 2.1, all the values of a and r within these ranges are bounded.

On the other hand, if equation (1.9) is applied to the predictor formula, we have

$$\xi^2 = \frac{1 + (2-a)q}{1 - aq} = \frac{y_{n+2}}{y_n}$$

however, since R(q) gives the stability region as $a > 0$ and $r < \frac{2}{3}$ then for $0 < a < 1$, $q = -0.125$, zero stability is satisfied since $|\xi| < 1$ for all a.

Furthemore, the combined Predictor-Corrector formula (2.13), substituted into (1.9) gives

$$\xi^2 = \frac{rq\left(\dfrac{1+(2-a)q}{1-aq}\right)^{\frac{3}{2}} + 1 + \frac{1}{2}q(2+r)}{1-\frac{1}{2}q(2-3r)}$$

thus, for $r < \frac{2}{3}$ and $a > 0$, we have for the range $a \varepsilon (1,2)$ and $r \varepsilon (-0.78, 1.99)$ that $|\xi| < 1$;

for example, when $a = 1.8$, $r = -1.6$ we obtain a complex value

$$\xi = -0.187i + 0.684$$

for which $|\xi| = 0.709 < 1$

This is true for all value pairs of a and r, hence the method is absolutely stable for all choices of free parameters a and r of the Predictor-Corrector method (2.13).

2.4 DERIVATION OF ORDER 3 SCHEME

The derivation of Predictor-Corrector integration formula of order 3 , for the class of MLMM being used varies. This is due to the maximum order required, thereby limiting the number of equations available for the determination of coefficients in equation (2.2) and (2.3).

For the Predictor formula given by (2.2), set $\alpha_0 = 1$, $\alpha_1 = \beta_1 = \gamma_1 = 0$ and let $\beta_2 = a$ (free parameter).

Fitting the Predictor formula to second order formula, we obtain,

$$\alpha_0 = -1$$

$$\beta_0 = 2 - a$$

and $\gamma_0 + \gamma_2 = 2 - 2a$

These lead to the choice

$$\gamma_0 = \gamma_2 = 1 - a$$

therefore (2.2) becomes

$$y_{n+2} - y_n = h\,[(2-a)f_n + af_{n+2}] + h^2[(1-a)g_n + (1-a)g_{n+2}]$$

$$\Rightarrow \quad \frac{y_{n+2}}{y_n} = \frac{1 + (2-a)q + (1-a)q^2}{1 - aq - (1-a)q^2}$$

$$= \frac{(q-1)[(a-1)q - 1]}{(q+1)[(a-1)q + 1]} = R^*(q) \qquad (2.14)$$

when a=1, (2.14) may not satisfy the stability requirements because

$$\lim_{q \to -\infty} \left| \frac{y_{n+2}}{y_n} \right| = 1$$

However, by choosing

$$\gamma_0 = 1 - \tfrac{1}{2}a \quad \text{and} \quad \gamma_2 = 1 - \tfrac{3}{2}a$$

together with other coefficients obtained, we have equation (2.2) becoming

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (1-\frac{1}{2}a)q^2}{1 + aq - (1-\frac{3}{2}a)q^2} = \bar{R}\,(q) \qquad (2.15)$$

evaluating a from this we shall obtain

$$a = \frac{(1-e^{2q})/q + 2 + q(e^{2q} + 1)}{1 - e^{2q} + \frac{1}{2}q(1+3e^{2q})} \qquad (2.16)$$

For the composite Corrector formula for case k=2 from equation (2.3) we set

$$c_0 = c_1 = c_2 = c_3 = 0$$

since $\alpha_2 = 1$, $\beta_3 = r$ (free parameter)

then, for $\alpha_1 = \beta_1 = \gamma_1 = 0$, we solve system of equation (2.4) to get

$$\alpha_0 = -1, \quad \beta_0 + \beta_2 = 2 - r$$

$$\gamma_0 = \frac{1}{3} - \frac{25}{4} r, \quad \gamma_2 = -\frac{1}{3} + \frac{5}{4} r$$

choosing $\beta_0 = 1 - 2r$ and $\beta_2 = 1 + r$

then (2.3) becomes

$$y_{n+2} - y_n = h[(1-2r)f_n + (1+r)f_{n+2} + rf_{n+3}] + h^2[(\tfrac{1}{3} - \tfrac{25}{4}r)g_n + (-\ \tfrac{1}{3} + \tfrac{5}{4}r)g_{n+2}]$$

using exponential fitting to get

$$r = \frac{(e^{2q} - 1)/q - (1 + e^{2q}) + \frac{1}{3}q(e^{2q} - 1)}{e^{3q} + \frac{1}{4}(4+5q)e^{2q} - \frac{1}{4}(8 + 15q)} \qquad (2.17)$$

and

$$\frac{y_{n+2}}{y_n} = \frac{4rqR^*(q) - rq(8+15q) + 4(1+q+\frac{1}{3}q^2)}{4(1-q+\frac{1}{3}q^2) - rq(4+5q)}$$

$$= \bar{R}(q). \quad \text{say} \qquad (2.18)$$

where

$$R^*(q) = \frac{\bar{y}_{n+3}}{y_n} = \left(\frac{\bar{y}_{n+2}}{y_n}\right)^{\frac{3}{2}} = [\ \bar{R}(q)\ ]^{\frac{3}{2}}$$

is given by step A3 of the CIF.

To examine the stability conditions required by this method, it is expected by Theorem 2.1 that (2.18) satisfies $|\bar{R}(q)| < 1$.

However, since equation (2.15) is contained in (2.18), then (2.15) must also satisfy $|\bar{R}(q)| < 1$.

From (2.18), $R(q) - 1 < 0$, as $q \to -\infty$ gives

$$\frac{\frac{4r}{q} R^*(q) - 10r}{\frac{4}{3} - 5r} < 0$$

that is, $r < \frac{4}{15}$.

Furthermore, from (2.15) examine

$$\bar{R}(q) - 1 < 0 \quad \text{as} \quad q \to -\infty$$

and after tediuos algebra, we obtain

$$\frac{2 - 2a}{-1 + \frac{3}{2}a} < 0$$

$\Rightarrow \quad a < \frac{2}{3} \quad$ or $\quad a > 1$

Also, taking $|R(q)| < 1$ as $-1 < R(q) < 1$,

then for $R(q) > -1$ we have $a > \frac{2}{3}$ or $a < 0$.

However from (2.16) the range of values of a for $q \in (-\infty, 0)$

taking the limits with L'Hospital rule we have

$$\lim_{q \to 0} a = \frac{2}{3} \quad \text{while} \quad \lim_{q \to -\infty} a = 2$$

that is, $a \in (\frac{2}{3}, 2)$.

Thus, as q decreases, a is monotone increasing.

Also, from (2.17)

$$\lim_{q \to 0} r = 1 \quad \text{and} \quad \lim_{q \to -\infty} r = \frac{4}{45}$$

That is for all values of $q < 0$ parameters r lies in the interval $(\frac{4}{45}, 1)$.

Thus as q increases, r also is monotone increasing.

For the ranges of a and r obtained, they suggest that the integration formula (2.18) may be A-stable. However, we established from algebra work done on the condition that $R(q)$ satisfy with ranges of values of a and r, that is within the ranges

$$a \in (1,2) \quad \text{and} \quad r \in (\frac{4}{45}, \frac{4}{15})$$

our integration formula (2.18) will satisfy the A-stability conditions given by the Maximum Modulus Theorem 2.1.

Alternative to the order three formulas (2.15) and (2.18) obtained above is by introducing two free parameters. The reason being that the choice of γ_0 and γ_2 for the predictor formula (2.15) and the choice made for β_0 and β_2 as it affects the corrector formula will go a long way to affect the stability and the accuracy of the Predictor-Corrector scheme (2.18). This will also affect the free parameters a and r. Thus by introducing two free parameters each into the predictor and the corrector formulas, a new set of formulas are obtained.

For the Predictor formula, set

$$\alpha_1 = \beta_1 = \gamma_1 = 0 \;, \qquad \alpha_2 = +1$$

and let $\beta_2 = a$ and $\gamma_2 = b$ (free parameters)

then, $\beta_0 = 2 - a$, $\alpha_0 = -1$

$$\gamma_0 = 2 - 2a - b$$

which gives the Predictor formula as

$$y_{n+2} - y_n = h\,[(2-a)f_n + af_{n+2}] + h[(2-2a-b)g_n + bg_{n+2}]$$

using exponential fittings we have

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (2-2a-b)q^2}{1 - aq - bq^2} = \bar{R}\,(q) \qquad (2.19)$$

In a similar manner, for corrector formula, set

$\beta_3 = r$, $\beta_2 = s$ as free parameters;

then for an order 3 corrector scheme in form of (2.3) we have

$$y_{n+2} - y_n = h[(2-s-r)f_n + sf_{n+2} + rf_{n+3}]$$
$$+ h^2[(\tfrac{4}{3} - s - \tfrac{3}{4}r)g_n + (\tfrac{2}{3} - s - \tfrac{9}{4}r)g_{n+2}]$$

on using (1.8) we obtain

$$\frac{y_{n+2}}{y_n} = \frac{1 + (2-s-r)q + (\frac{4}{3} - s - \frac{3}{4}r)q + rq\,(y_{n+3}/y_n)}{1 - sq - q(\frac{2}{3} - s - \frac{9}{4}r)} \qquad (2.20)$$

By step A3 of the Algorithm, we obtain

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \bar{R}\,(q) = \frac{y_{n+2}}{y_n}$$

or $$\frac{\bar{y}_{n+3}}{y_n} = \left(\frac{y_{n+2}}{y_n}\right)^{\frac{3}{2}} = R^*(q)$$

making corrector formula become

$$\frac{y_{n+2}}{y_n} = \frac{1 + (2-s-r)q + (\frac{4}{3} - s - \frac{3}{4}r)q^2 + rq.R^*(q)}{1 - sq - q(\frac{2}{3} - s - \frac{9}{4}r)}$$

$$= R(q) \qquad (2.21)$$

Now using exponential fitting on (2.19), we obtain

$$(e^{2q} - 1) - 2q(q+1) = aq(e^{2q} - 1 - 2q) + bq^2(e^{2q} - 1) \qquad (2.22)$$

For the two unknowns a and b, this formula is fitted to two values of q as q_0 and q_1.

Thus let $C_i = e^{2q} - 1 - 2q_i(q_i+1)$

$A_i = q_i(e^{2q} - 1 - 2q_i)$

$B_i = q_i^2(e^{2q} - 1)$

then (2.22) can be written as

$$C_i = aA_i + bB_i \qquad i = 0,1$$

leading to two equations

$$C_0 = aA_0 + bB_0$$

$$C_1 = aA_1 + bB_1$$

in two unknowns. Solving for a and b we obtain

$$a = \frac{B_1C_0 - B_0C_1}{B_1A_0 - B_0A_1}$$

$$b = \frac{A_1C_0 - A_0C_1}{B_1A_0 - B_0A_1}$$

Similarly, the corrector formula (2.20) can be written as

$$e^{2q}(1 - \tfrac{2}{3}q^2) - 1 - 2q - \tfrac{4}{3}q^2 = s[q(e^{2q} - 1) - q^2(e^{2q} + 1)]$$

$$+ r[q(e^{3q} - 1) - \tfrac{3}{4} q^2(3e^{2q} + 1)$$

If $T_i = e^{2q}(1 - \frac{2}{3} q_i^2) - 1 - 2q - \frac{4}{3} q_i^2$

$S_i = q_i(e^{2q} - 1) - q_i(e^{2q} + 1)$

$R_i = q_i(e^{3q} - 1) - \frac{3}{4}q_i^2(3e^{2q} +1)$

then

$$T_i = sS_i + rR_i \qquad i = 0,1$$

or

$$T_0 = sS_0 + rR_0$$

$$T_1 = sS_1 + rR_1$$

solving to get

$$s = \frac{R_1T_0 - R_0 T_1}{R_1S_0 - R_0 S_1} \quad \text{and} \quad r = \frac{S_1T_0 - S_0T_1}{S_1R_0 - S_0R_1}$$

To investigate the behaviour of the parameters a, b, s and r, substitute (2.21) into the inequality

$$R(q) - 1 < 0$$

After much algebra we shall obtain

$$s > -\frac{1}{6} \quad \text{and} \quad s + \frac{9}{4} r - \frac{2}{3} > 0$$

with $R^*(q) < 1$ which by (2.19) we have

$$\bar{R}(q) < 1 \quad \text{or} \quad \frac{2(a-1)}{b} < 0$$

If $b > 0$, then $a > 1$.

For all real roots of the quadratic expression of the numerator of (2.19), we have

$$(a+2)^2 > 4(2-b)$$

and its denominator gives

$$a^2 + 4b > 0 \implies b > 0$$

In conclusion, the behaviour of these parameters along with the analytical conclusion above is best done by keeping q_1 fixed and q_0 is varied in the interval $(-\infty, 0]$ to satisfy stiff problem (1.8).

2.5 DERIVATION OF ORDER 4 SCHEME

Deriving a second derivative Predictor formula of order 3 corresponding to (2.5) which also contains a free parameter, deduce from (2.4) that

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + 2\alpha_2 - \beta_0 - \beta_1 - \beta_2 = 0$$

$$\frac{1}{4}(\alpha_1 + 4\alpha_2) - (\beta_1 + 2\beta_2) - (\gamma_0 + \gamma_1 + \gamma_2) = 0$$

$$\frac{1}{6}(\alpha_1 + 8\alpha_2) - \tfrac{1}{2}(\beta_1 + 4\beta_2) - (\gamma_1 + 2\gamma_2) = 0$$

Obtaining a two step formula, we set $\alpha_1 = \beta_1 = \gamma_1 = 0$ and let $\beta_2 = a$ (free parameter), then

$$\alpha_2 = 1, \quad \alpha_0 = -1, \quad \beta_0 = 2 - a, \quad \gamma_0 = \frac{4}{3} - a, \text{ and } \gamma_2 = \tfrac{2}{3} - a$$

Thus the predictor method is

$$y_{n+2} - y_n = h[(2-a)y_n' + ay_{n+2}'] + h^2[(\tfrac{4}{3} - a)y_n'' + (\tfrac{2}{3}-a)y_{n+2}''] \qquad (2.24)$$

Using (1.8), we have

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (\frac{4}{3} - a)q^2}{1 - aq - (\frac{2}{3} - a)q^2} = \bar{R}(q) \qquad (2.25)$$

and by (1.8) and (2.7), equation (2.24) can be written as

$$a = \frac{1 + 2q + \frac{4}{3}q^2 + \frac{1}{3}e^{2q}(2q^2 - 3)}{qe^{2q}(q-1) + q(q+1)} \qquad (2.26)$$

The Corrector formula of order 4 corresponding to (2.3) gives the set of coefficient equations to be determined as follows:

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + 2\alpha_2 - \beta_0 - \beta_1 - \beta_2 - \beta_3 = 0$$

$$\tfrac{1}{2}(\alpha_1 + 4\alpha_2) - (\beta_1 + \beta_2 + 3\beta_3) - (\gamma_0 + \gamma_1 + \gamma_2) = 0$$

$$\frac{1}{6}(\alpha_1 + 8\alpha_2) - \tfrac{1}{2}(\beta_1 + 4\beta_2 + 9\beta_3) - (\gamma_1 + 2\gamma_2) = 0$$

$$\frac{1}{24}(\alpha_1 + 16\alpha_2) - \frac{1}{6}(\beta_1 + 8\beta_2 + 27\beta_3) - \tfrac{1}{2}(\gamma_1 + 4\gamma_2) = 0$$

setting $\alpha_1 = \beta_1 = \gamma_1 = 0$, $\alpha_2 = 1$ and the free parameter $\beta_3 = r$ we obtain other coefficients as

$\alpha_0 = -1$, $\beta_0 = 1-r$, $\beta_2 = 1$, $\gamma_0 = \frac{1}{3} - \frac{3}{4}r$ and $\gamma_2 = -\frac{1}{3} - \frac{9}{4}r$

Therefore the composite Corrector Integration Formula (2.3) becomes

$$y_{n+2} - y_n = h\,[(1-r)y'_n + y'_{n+2} + ry'_{n+3}] + h\,[(\frac{1}{3} - \frac{3}{4}r)y''_n - (\frac{1}{3} + \frac{9}{4}r)y''_{n+2}]$$

using the scalar test problem (1.8), we obtain

$$\frac{y_{n+2}}{y_n} = \frac{rq.(\bar{y}_{n+3}/y_n) + 1\ (1-r)q + (\frac{1}{3} - \frac{3}{4}r)q^2}{1 - q + (\frac{1}{3} - \frac{9}{4}r)q^2} \tag{2.27}$$

which can also be written for r as

$$r = \frac{1 + q + \frac{1}{3}q^2 - e^{2q}(1 - q + \frac{1}{3}q^2)}{\frac{3}{4}q^2(3e^{2q} + 1) - q(e^{3q}-1)} \tag{2.28}$$

To unite both the Predictor and the Corrector formulae, we proceed as before on steps A3 through A5 of the CIF. From previous derivation, it has been established that our set of integration formulas are consistent, that is

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \frac{\bar{y}_{n+2}}{y_n} = \bar{R}(q)$$

leading to

$$\frac{\bar{y}_{n+3}}{y_n} = \left(\frac{\bar{y}_{n+2}}{y_n} \right)^{\frac{3}{2}} = R^*(q)$$

Finally from step A5 of the Algorithm, (2.27) becomes

$$\frac{y_{n+2}}{y_n} = \frac{rqR^*(q) +1 + (1-r)q + (\frac{1}{3} - \frac{3}{4} r)q^2}{1 - q + (\frac{1}{3} + \frac{9}{4} r)q^2} \tag{2.29}$$

Equation (2.29) is the Composite Integration Formula of order 4 for stiff problems which allow exponential fittings.

For stability conditions which R(q) must satisfy, we check by analytical processes for

$$R(q) < 1$$

And from (2.29) we have

$$\frac{rqR^*(q) + (1+r)q - 3rq^2}{1 - q + (\frac{1}{3} + \frac{9}{4} r)q^2} < 0$$

taking limit as $q \to -\infty$, we have

$$\frac{\lim_{q \to -\infty} \frac{r}{q} R^*(q) - 3r}{\frac{1}{3} + \frac{9}{4} r} < 0$$

this gives $r > 0$ or $r < -\frac{4}{27}$

however, we need to check the conditions satisfied by a,

that is, $R(q) - 1 < 0$ as $q \to -\infty$

from (2.27), we have

$$a < \tfrac{2}{3} \quad \text{or} \quad a > 1$$

hence, when $r > 0$, then $\lim_{q \to -\infty} \frac{r}{q}.R^*(q)$ required that $a > 1$

However, parameters a and r are bounded by certain limiting values for $q \varepsilon (-\infty, 0)$ Thus as $q \to 0$ and $q \to -\infty$ simultaneously, we have from (2.27)

$$\lim_{q \to 0} a = 1 \qquad \lim_{q \to -\infty} a = \frac{4}{3}$$

similarly, $$\lim_{q \to 0} r = \frac{16}{135} \qquad \lim_{q \to -\infty} r = \frac{4}{9}$$

that is $a \varepsilon (1, \frac{4}{3})$ while $r \varepsilon (\frac{16}{135}, \frac{4}{9})$.

Hence, we established analytically for stiff system whose solutions are in the open left half plane,

that is $a > 1$ and $r > 0$

Furthermore, evaluating the values of a and r for some sample S, we obtain table 2.2. That is as q decreases, parameters a and r are monotone increasing, as given below:

TABLE 2.2
Values of parameters a and r for order 4 scheme

q	a	r
-1	1.0648	0.16829
-2	1.1204	0.21710
-3	1.16297	0.25855
-20	1.30088	0.40833
-50	1.3201	0.42977
-100	1.3267	0.43707

By Theorem 2.1, we conclude that our integration formula (2.29), which is a MLMM of order 4, is A-stable within the range of values specified for the choices of parameters a and r.

The stability polynomial used to determine the zero- and absolute-stabilities of the methods revealed that by equation (1.9), the predictor formula (2.25) gives

$$\xi = \frac{1 + (2-a)q + (\frac{4}{3} - a)q^2}{1 - aq - (\frac{2}{3} - a)q^2}$$

Evaluating for all values of $a \varepsilon (1 , \frac{4}{3})$ in the left-half plane, we obtain

$$|\xi| < 1 \quad \text{for all} \quad q \varepsilon (-\infty, 0)$$

Thus, in general for all $q \varepsilon (-\infty, 0)$ the predictor scheme (2.25) is zero-stable.

Also the stability polynomial for the Predictor-Corrector formula following (1.9) gives

$$\xi^2 = \frac{rq^2R^*(q) + 1 + (1-r)q + (\frac{1}{3} - \frac{3}{4}r)q^2}{1 - q + (\frac{1}{3} - \frac{9}{4}r)q^2}$$

Testing for values of $q \varepsilon (-\infty, 0)$ we established that

$$|\xi| < 1 \quad \text{for all } q$$

that is , the Predictor-Corrector formula fitted to stiff scalar problem (1.8) is absolutely stable.

This order 4 formula tested on various problems is seen to be more accurate than many known methods for the same problems. This shall be illustrated in chapter 4.

CHAPTER THREE

PADE EXPONENTIAL FORMULAS

3.1 INTRODUCTION

The derivative of stiff integration formula s of orders higher than order 4 will involve a one step ratio y_{n+1}/y_n which can be replaced either by an exponential function or by a one step scheme.

Usually the predictor formula involving this ratio is implicit. Hence to retain two step implicit scheme, the ratio y_{n+1}/y_n in the predictor formula is replaced not with an exponential function, but with an one step implicit method expressed as a 1-1 Pade approximation function. However, since the formula given in this thesis is designed for stiff problems for which exponential fitting is permitted, the exponential function will also be an appropriate substitution for the ratio y_{n+1}/y_n.

3.2 ORDER 5 INTEGRATION FORMULA

The order 5 formula introduces a one-step scheme at f_{n+1} term. This is chosen so that two-step procedure is maintained on the left hand side of equation (2.2). Hence, α_1 and γ_1 are set to zero in the simultaneous equations required to fix the fifth order integration formula.

Fitting order 4 predictor scheme using (2.4) for $c_j = 0$, $j = 0,1,2,3,4$ we obtain

$$\alpha_0 = -1 , \qquad \alpha_1 = 0 , \qquad \alpha_2 = 1$$

$$\beta_0 = a \qquad \beta_1 = 2(1-a) \qquad \beta_2 = a \quad \text{(free parameter)}$$

$$\gamma_0 = -\frac{1}{6}(1-2a) , \qquad \gamma_1 = 0 , \quad \gamma_2 = \frac{1}{6}(1-2a)$$

Hence by (2.2) we obtain the predictor formula as

$$y_{n+2} - y_n = h\,[af_n + 2(1-a)f_{n+1} + af_{n+2}] + \frac{1}{6}h^2\,[(1-3a)g_{n+2} - (1-3a)g_n]$$

By (1.8) we obtain

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + aq - \frac{1}{6}(1-3a)q^2 + 2(1-a)q.(y_{n+1}/y_n)}{1 - aq - \frac{1}{6}(1-3a)q^2} \tag{3.1}$$

The one-step y_{n+1}/y_n term is replaced by an exponentially fitted one-step implicit scheme. For this purpose, consider the most accurate one step implicit scheme, namely, the Mid-point rule given by

$$y_{n+1} - y_n = \tfrac{1}{2}h\,(f_n + f_{n+1})$$

using (1.8) we obtain a 1-1 Pade approximation

$$\frac{y_{n+1}}{y_n} = \frac{1 + \frac{1}{2}q}{1 - \frac{1}{2}q} \tag{3.2}$$

substituting in (3.1), we have

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (\frac{3}{2} - a)q + (\frac{5}{6} - a)q^2 + \frac{1}{12}(1 - 3a)q^3}{1 - (\frac{1}{2} + a)q + (a - \frac{1}{6})q^2 + \frac{1}{12}(1 - 3a)q^3}$$

$$= \bar{R}(q)$$

However, replacing the ratio y_{n+1}/y_n in (3.1) by e^q according to the procedure in the previous chapter, we have

$$\frac{y_{n+2}}{y_n} = \frac{1 + aq - \frac{1}{6}(1-3a)q^2 + 2(1-a)qe^q}{1 - aq - \frac{1}{6}(1-3a)q^2}$$

Furthermore, from (3.1), we have

$$a = \frac{-1 + \frac{1}{6}q^2 - 2qe^q + (1 - \frac{1}{6}q^2)e^{2q}}{q(1+\frac{1}{2}q) - 2qe^q + q(1-\frac{1}{2}q)e^{2q}} \tag{3.3}$$

Similarly, to derive an order 5 corrector formula, set the following unknowns in composite formula (2.3) as

$\alpha_1 = \gamma_1 = 0$, $\beta_3 = r$ (free parameter)

$\alpha_2 = +1$, $\alpha_0 = -1$

On solving the system generated by $c_j = 0$, $j = 0,1,\ldots,5$

we obtain

$$\beta_0 = \frac{7}{15} + \frac{7}{2}r \qquad \beta_1 = \frac{16}{15} - 9r$$

$$\beta_2 = \frac{7}{15} + \frac{9}{2}r \qquad \gamma_0 = \frac{1}{15} + \frac{3}{2}r \qquad \gamma_2 = -\frac{1}{15} - 9r$$

Hence the Corrector Integration Formula is given by

$$y_{n+2} - y_n = h\,[(\frac{7}{15} + \frac{7}{2}r)f_n + (\frac{16}{15} - 9r)f_{n+1} + (\frac{7}{15} + \frac{9}{2}r)f_{n+2} + rf_{n+3}] + h^2\,[(\frac{1}{15} + \frac{3}{2}r)g_n + (-\frac{1}{15} - \frac{9}{2}r)g_{n+2}]$$

And with exponential fittng, it becomes

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{7}{15} + \frac{7}{2}r)q + (\frac{1}{15} + \frac{3}{2}r)q^2 + (\frac{16}{15} - 9r)q.(\bar{y}_{n+1}/y_n) + rq(\bar{y}_{n+3}/y_n)}{1 - (\frac{7}{15} + \frac{7}{2}r)q + (\frac{1}{15} + \frac{9}{2}r)q^2} \tag{3.4}$$

with localising assumption, we obtain r from (3.4) as

$$r = \frac{1 + \frac{7}{15}q + \frac{1}{15}q^2 + \frac{16}{15}qe^q - (1 - \frac{7}{15}q + \frac{1}{15}q^2)e^{2q}}{-\frac{7}{2}q - \frac{3}{2}q^2 + 9qe^q + \frac{9}{2}qe^{2q}(q-1) - qe^{3q}}$$

From previous derivation, we established that step A3 of the algorithm gives

$$\frac{\bar{y}_{n+3}}{y_{n+1}} = \frac{\bar{y}_{n+2}}{y_n} = \bar{R}(q)$$

which leads to

$$\frac{\bar{y}_{n+3}}{y_n} = [\bar{R}(q)]^{\frac{3}{2}} = R_2(q)$$

and (3.6)

$$\frac{\bar{y}_{n+1}}{y_n} = [\bar{R}(q)]^{\frac{1}{2}} = R_1(q)$$

Thus instead of introducing another pade approximation, we simply fit the predictor formula, which has an 'in-built' pade approximation, into (3.5) by using (3.6).

Hence the Predictor-Corrector formula of order 5 is

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{7}{15} + \frac{7}{2}r)q + (\frac{1}{15} + \frac{3}{2}r)q^2 + (\frac{16}{15} - 9r)q.R_1(q) + rq.R_2(q)}{1 - (\frac{7}{15} + \frac{9}{2}r)q + (\frac{1}{15} + \frac{9}{2}r)q^2} \qquad (3.7)$$

As it is done for lower order formulas, we obtain the range of values of a and r for which $q \varepsilon (-\infty, 0)$

Now, $\lim\limits_{q \to -\infty} a = 0$

To obtain the limiting value of a as $q \to 0$, we apply L'Hospital rule at five stages involving tedious and careful differentiations to give

$$\lim_{q \to 0} a = \frac{19}{27}$$

that is for $q \varepsilon (-\infty \; 0)$, $a \varepsilon (0, \frac{19}{27})$

similarly, for $q \varepsilon (-\infty \; 0)$, $r \varepsilon (-\frac{2}{45}, 0)$

The values of a and r for some samples S in the range q are given in Table 3.1 below:

TABLE 3.1

q	a	r
-1	0.464189	-0.001883
-10	0.39168	-0.026082
-20	0.36481	-0.034591
-30	0.35476	-0.037751
-40	0.34956	-0.039381
-50	0.34639	-0.040373
-1000	0.33399	-0.044237

These results, as it was done in lower orders guarantee A-Stability of the order 5 scheme.

3.3 ORDER 6 INTEGRATION FORMULA

Following the procedure for order 5, the Predictor formula is obtained by solving six simultaneous equations corresponding to equation (2.4) to give

$$y_{n+2} - y_n = h\,[(\tfrac{14}{15} - a)f_n + \tfrac{16}{15}f_{n+1} + af_{n+2}] + h^2[(\tfrac{2}{9} - \tfrac{1}{3}a)g_n + \tfrac{4}{3}(\tfrac{7}{15} - a)g_{n+1} + (\tfrac{4}{45} - \tfrac{1}{3}a)g_{n+2}] \tag{3.8}$$

Similarly, choosing $\beta_3 = r$ as the free parameter, we obtain the Corrector formula by equation (2.4) as

$$y_{n+2} - y_n = h\,[(\tfrac{7}{15} - 10r)f_n + (\tfrac{16}{15} - 9r)f_{n+1} + (\tfrac{7}{15} + 18r)f_{n+2} + rf_{n+3}] + h^2[(\tfrac{1}{15} - 3r)g_n - 18rg_{n+1} - (\tfrac{1}{15}+9r)g_{n+2}] \tag{3.9}$$

On fitting to the scalar problem (1.8), the Predictor formula (3.8) becomes

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (\frac{14}{15} - a)q + (\frac{2}{9} - \frac{1}{3}a)q^2 + [\frac{16}{15}q + (\frac{28}{45} - \frac{4}{3}a)q^2].\frac{y_{n+1}}{y_n}}{1 - aq - (\frac{4}{45} - \frac{1}{3}a)q^2} \tag{3.10}$$

using a 1-1 pade approximation (3.2), we obtain

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (\frac{3}{2} - a)q + (\frac{41}{45} - \frac{7}{6}a)q^2 + (\frac{1}{5} - \frac{1}{2}a)q^3}{1 - (\frac{1}{2} + a)q - (\frac{4}{45} - \frac{5}{6}a)q^2 + (\frac{2}{45} - \frac{1}{6}a)q^3}$$

$$= \bar{R}(q)$$

Replace the ratio y_{n+1}/y_n by e^q in (3.10) the sixth order Predictor formula can be expressed as

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{14}{15} - a)q + (\frac{2}{9} - \frac{1}{3}a)q^2 + [\frac{16}{15}q + (\frac{28}{45} - \frac{4}{3}a)q^2]e^q}{1 - aq - \frac{1}{3}(\frac{4}{45} - a)q^2}$$

obtaining parameter a from (3.10), we have

$$a = \frac{1 + \frac{14}{15}q + \frac{2}{3}q^2 + \frac{4}{15}qe^q(12+7q) + \frac{1}{15}e^{2q}(4q^2 - 45)}{q(3+q) + 4q^2e^q - qe^{2q}(3-q)} \tag{3.11}$$

In a similar manner we fit (1.8) into (3.9) to get the corrector integration scheme as

$$\frac{y_{n+2}}{y_n} = \frac{1 + (\frac{7}{15} - 10r)q + (\frac{1}{15} - 3r)q^2 + [(\frac{16}{15} - 9r)q - 18rq^2].[\bar{R}(q)]^{\frac{1}{2}} + rq.[\bar{R}(q)]^{\frac{3}{2}}}{1 - (\frac{7}{15} + 18r)q + (\frac{1}{15} + 9r)q^2} \tag{3.12}$$

and obtaining r from (3.9) we have

$$r = \frac{1 + \frac{7}{15}q + \frac{1}{15}q^2 + \frac{16}{15}qe^q - e^{2q}(1 - \frac{7}{15}q + \frac{1}{15}q^2)}{q(10+3q) + 9qe^q(1+2q) - 9qe^{2q}(2-q) - qe^{3q}} \tag{3.13}$$

To obtain the range of values of a and r for which the order 6 Predictor -Corrector formula may be A-Stable, we apply L'Hospital rule for six derivatives on equation (3.11) to get

$$\lim_{q \to 0} a = \frac{7}{15}$$

and $$\lim_{q \to -\infty} a = \frac{2}{3}$$

Similarly, 7 times application of L'Hospital rule on (3.13) gives

$$\lim_{q \to 0} r = \frac{4}{945}$$

while $$\lim_{q \to -\infty} r = \frac{1}{45}$$

hence for $q \varepsilon (-\infty, 0)$ we have $a \varepsilon (\frac{7}{15}, \frac{2}{3})$ and $r \varepsilon (\frac{4}{945}, \frac{1}{45})$.

These show that as q increases, both a and r decreases. Following previous work, we established that within these ranges of values of a and r, the Predictor-Corrector formula (3.12) will be A-Stable for all choices of parameters a and r.

3.4 ORDER 7 SCHEME

Finally as previously stated that the highest attainable order for the second derivative LMM is seven. We hereby derive an order 7 Predictor-Corrector formula. The formula is derived without any free parameter as this is not required by the formula of this order.

However, obtaining a predictor formula of order 6 from (2.4), we obtain

$$y_{n+2} - y_n = \frac{h}{15}(7f_n + 16f_{n+1} + 7f_{n+2}) + \frac{h^2}{15}(g_n - g_{n+2})$$

Similarly, for order 7 Corrector scheme, eight equations for $c_j = 0$, $j = 0,1,\ldots,7$ obtained from (2.4) are solved to give

$$y_{n+2} - y_n = \frac{h}{35}\left(\frac{1049}{54} f_n + 34f_{n+1} + \frac{33}{2} f_{n+2} + \frac{2}{27} f_{n+3}\right) + \frac{h^2}{105}\left(\frac{37}{3} g_n + 4g_{n+1} - 8g_{n+2}\right) \qquad (3.14)$$

Equation (3.14) is the composite corrector formula.

Using exponential fitting, the predictor formula gives

$$\frac{\bar{y}_{n+2}}{y_n} \qquad \frac{15 + 7q + q^2 + 16q.(y_{n+1}/y_n)}{15 - 7q + q^2}$$

Using 1-1 pade approximation, we obtain

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{30 + 31q + 11q^2 - q^3}{30 - 29q + 9q^2 - q^3} \tag{3.15}$$

$$= \bar{R}(q)$$

substituting (3.15) into the corrector scheme (3.14) after using exponential fitting, we obtain the Predictor-Corrector formula of order 7 as

$$\frac{y_{n+2}}{y_n} = \frac{1 + \frac{1049}{1890} q + \frac{37}{315} q^2 - (\frac{34}{35} q + \frac{4}{105} q^2).[\bar{R}(q)]^{\frac{1}{2}} + \frac{2}{945} [\bar{R}(q)]^{\frac{3}{2}}}{1 - \frac{33}{70} q + \frac{8}{105} q^2}$$

These formulas derived so far in chapters 2 and 3 are coded in fortran language to solve several system of Ordinary Differential Equations, among which are the standard stiff Initial Value Problems discussed in the next chapter.

CHAPTER FOUR

COMPARATIVE ANALYSIS

4.1 INTRODUCTION

The formulas derived so far in the last two chapters are tested on several standard Stiff Initial Value Problems (IVPs) [2,6,7,8,11,16]. The aim of the computational analysis carried out in this chapter is firstly to show how these composite formulas compare fovourably with other known methods and the accuracy of these formulas when compared with the exact solution. Secondly, to demonstrate the performance of the step control procedure. For system of stiff IVPs, the eigenvalues are obtained from the Jacobian matrix of the system.

All computations given in this thesis are coded in a fortran 77 on a 12/286MX, Leading Technology (IBM compatible) computer of Lagos State University, Ojo; using double precision arithmetic.

4.2 NUMERICAL EXAMPLES

PROBLEM 1

Consider the system of stiff IVP

$$\begin{aligned} y' &= -y + 95z, \quad & y(0) &= 1 \\ z' &= -y - 97z, \quad & z(0) &= 1 \end{aligned} \qquad x \in [0,1] \qquad (4.1)$$

Attempt will be made to demonstrate the efficiency of the formulas derived in chapters 2 and 3. Also given is the comparison of accuracy with other exponentially fitted schemes proposed by Kenue and Jackson [17] and Cash [2].

The eigenvalues of the Jacobian matrix of system (4.1) are

$\lambda_1 = -2$ and $\lambda_2 = -96$

and the general solution is of the form

$$\begin{aligned} y(x) &= A\, e^{\lambda_1 x} + B\, e^{\lambda_2 x} \\ z(x) &= C\, e^{\lambda_1 x} + D\, e^{\lambda_2 x} \end{aligned} \tag{4.2}$$

however, imposing another initial conditions at the derivatives of y and z, we obtain from (4.1)

$$y'(0) = 94$$

$$z'(0) = -98$$

and $e^{\lambda_2} \to 0$ as $x \to 0$

leaving us with $A = \frac{95}{47}$ and $c = \frac{-1}{47}$

Problem (4.1) is solved and tested on all our newly developed schemes of orders 2 to 7 using various step lengths.

For example, by using a step length $h = 0.0625$, we obtain $q = -0.125$ because $q = \lambda_1 h$.

The procedure for obtaining a result required the evaluation of parameters a and r and the quantity $R(q)$. The program to solve this problem along with various output for orders two to seven are given in Appendix A.

Denote the second derivative two-step composite schemes derived in chapter 2 for orders 2 through 4 by SS2, SS3 and SS4 respectively. Furthermore, denote orders 5,6 and 7 formulas involving pade-approximations by PS5, PS6 and PS7, while those involving exact exponential fittings are denoted by ES5, ES6 and ES7 respectively.

Some results for the solution of problem 1 which are extracted from Appendix A are given in Table 4.1(a) below:

TABLE 4.1(a)

Accuracy table for MLMM on Problem 1

step	method	max error y	max error z
0.03125	SS4	2.5×10^{-11}	2.6×10^{-13}
0.0625	SS2	5.6×10^{-17}	8.7×10^{-19}
	SS3	2.9×10^{-8}	3.1×10^{-10}
	SS4	4.4×10^{-16}	4.3×10^{-18}
	PS5	4.9×10^{-6}	5.2×10^{-8}
	ES5	5.6×10^{-17}	8.7×10^{-19}
	PS6	4.8×10^{-6}	4.8×10^{-8}
	PS7	6.0×10^{-3}	6.3×10^{-5}
	ES7	6.0×10^{-3}	6.3×10^{-5}
0.125	SS4	1.1×10^{-16}	8.7×10^{-19}
	ES5	1.1×10^{-16}	8.7×10^{-19}
0.25	SS2	0.0000	0.0000

The results above show that formulas SS2, SS4 and ES5 gave the least error for this problem. 1-1 Pade-approximation integrated into the formulas of orders 5, 6 and 7 affected the accuracy of their solutions, hence rather than using 1-1 Pade-approximation to estimate y_{n+1}/y_n, an exponential function e^q is rather preferred since it is exact for the exponential fittings. Hence ES5 gave much accuracy as orders 2 and 4 and identical to method SS4 for h = 0.125. Methods PS7 and ES7 performed poorly due to non existence of a free parameter which is an important factor in the derivation of other lower ordered schemes. Hence for a second derivative 2-step multistep methods, the maximal order 7 will perform poorly. However, orders 2 to 6 are sufficient to procure accurate solutions to any stiff problem for which exponential fitting is permitted.

We shall compare the results obtained by these methods with that of Cash [2] and Jackson and Kenue [17]. Let J-K denotes the method proposed by Jackson and Kenue, while SC4 and SC5 respectively denote the fourth and the fifth order schemes given by Cash for the case k = 1 of the MLMM. As shown in the Table 4.1(b) below, it will be observed that the methods proposed in this thesis are more accurate than the already existing ones.

TABLE 4.1(b)

Method	y(1)	z(1) x 10^{-2}	Error y	Error z
	$h = 0.0625$			
J-K	0.2725503	-0.2879477	$3x10^{-7}$	$4x10^{-9}$
SC4	0. 2735498	-0.2879471	$3x10^{-7}$	$3x10^{-9}$
SC5	0.27355005	-0.28794742	$1x10^{-8}$	$1x10^{-10}$
SS2	0.2735500406	-0.287947411	$6x10^{-17}$	$9x10^{-19}$
SS4	0.2735500406	-0.287947411	$3x10^{-16}$	$3x10^{-18}$
ES5	0.2735500405	-0.287947411	$6x10^{-17}$	$9x10^{-19}$
PS6	0.2735465656	-0.287943753	$4x10^{-6}$	$4x10^{-8}$
	$h = 0.03125$			
J-K	0.27355005	-0.28794742	$1x10^{-8}$	$1x10^{-10}$
SC4	0.27355003	-0.28794740	$1x10^{-8}$	$1x10^{-10}$
SS4	0.27355005	-0.287947402	$2x10^{-17}$	$2x10^{-19}$
True Sol.	0.2735500406	-0.287947411	---	---

PROBLEM 2 (4 x 4 Linear system)

Consider the 4 x 4 system of stiff Initial Value Problem

$$
\begin{aligned}
y_1' &= -10^4 y_1 + 100y_2 - 10y_3 + y_4 & y_1(0) &= 1 \\
y_2' &= -1000y_2 + 10y_3 - 10y_4 & y_2(0) &= 1 \\
y_3' &= -y_3 + 10y_4 & y_3(0) &= 1 \\
y_4' &= -0.1y_4 & y_4(0) &= 1
\end{aligned}
\tag{4.3}
$$

Putting in the matrix form, we obtain the eigenvalues of the Jacobian as

$$\lambda_1 = -0.1 \ , \quad \lambda_2 = -1 \ , \quad \lambda_3 = -1000 \ , \quad \lambda_4 = -10000$$

Following the arguement in Problem 1, the exact solution will be of the form

$$y_i(x) = v_i \, e^{\lambda_1 x} + w_i \, e^{\lambda_2 x}, \qquad i = 1,\ldots,4 \tag{4.4}$$

where

$$v_1 = \frac{-9909 - \lambda_2}{\lambda_1 - \lambda_2}$$

$$v_2 = \frac{-1000 - \lambda_2}{\lambda_1 - \lambda_2}$$

$$v_3 = \frac{-9 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$v_4 = \frac{-0.1 - \lambda_2}{\lambda_1 - \lambda_2}$$

$$w_1 = \frac{9909 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$w_2 = \frac{1000 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$w_3 = \frac{9 - \lambda_1}{\lambda_1 - \lambda_2}$$

$$w_4 = \frac{0.1 + \lambda_1}{\lambda_1 - \lambda_2}$$

The system (4.3) was considered by Enright and Pryce [8] and the error tolerance was fixed at 10^{-5}.

The order 4 scheme which gives a very high accuracy in Problem 1 is used to solve the stiff system (4.3) for $x \varepsilon [0,1]$.

Both versions of order 7 formula, that is PS7 and ES7, perform poorly as given in Appendix B. The error for y at x = 0.1 is poor with

$$y_1 = 20.0 , \quad y_2 = 2.0 , \quad y_3 = 0.04 , \quad y_4 = 0.04$$

However, the result obtained using the order 4 formula derived in Chapter 2 is given in Table 4.2 below.

TABLE 4.2

Efficiency of Order 4 Scheme

Step	$y_1(1)$	$y_2(1)$	$y_3(1)$	$y_4(1)$
0.05	-5910.942866	-595.6555	6.334079	0.90483742
0.1	-5910.942866	-595.6555	6.334079	0.90483742
True Sol.	-5910.942866	-595.6555	6.334079	0.90483742
ERROR				
0.05	4.5×10^{-12}	$4.5\text{x}10^{-13}$	$3.6\text{x}10^{-15}$	2×10^{-16}
0.1	3.6×10^{-12}	$3.4\text{x}10^{-13}$	$3.1\text{x}10^{-15}$	1×10^{-16}

The results obtained by order 4 show that the error tolerance can be raised to 10^{-12} as against 10^{-5} given by Enright and Pryce [8]

The results obtained at x = 1 for h = 0.05 involved 10 steps while for h = 0.1 required only 5 steps. However, the results obtained for h = 0.1 are more accurate.

PROBLEM 3 (SECOND ORDER DIFFERENTIAL EQUATION)

Dalquist and Bjock [6] showed that there are some stiff problems for which Runge-Kutta (R-K) method is unsuitable. One of such problems is a second order differential equation

$$\begin{aligned} y'' + 1001\,y' + 1000\,y &= 0 \\ y(0) = 1\,, \quad y'(0) &= -1 \end{aligned} \tag{4.5}$$

This problem has a general solution given by

$$y(x) = A\,e^{-x} + B\,e^{-1000x}$$

and for solution in [0,1] the exact solution is $y(x) = e^{-x}$. By setting $y' = z$, a 2×2 system of stiff IVP is obtained, that is

$$\begin{aligned} y' &= z \\ z' &= -1001\,z - 1000\,y \end{aligned} \tag{4.6}$$

with initial conditions $y(0) = 1$, $z(0) = -1$.

R-K method explodes for $h = 0.0027$ (approximately). Although this is unsatisfactory step size for describing the function e^{-x} however on trying it on the new methods much improved result is obtained. The performance of R-K method is partly due to its explicit nature which are generally unsuitable for stiff problems. The eigenvalues of the Jacobian matrix of (4.6) are $\lambda_1 = -1$, and $\lambda_2 = -1000$. This problem is solved (see Appendix C) using orders 3, 4 and 6 of the newly derived schemes and the results obtained at $x = 1$ is given in Table 4.3 below.

TABLE 4.3

Experimental Result on Second order ODE.

h	Formula	y(1)	Error
0.0027	R-K	0.367885	5.6×10^{-6}
	SS4	0.367879435	1.7×10^{-15}
0.05	SS3	0.367879435	2.0×10^{-10}
	SS4	0.367879435	1.7×10^{-16}
	PS6	0.367879436	5.6×10^{-8}
0.1	SS4	0.367879435	2.2×10^{-16}
	Exact Solution	0.367879435	----

It will be observed that for h = 0.0027 our new set of formulas including the less accurate order 6 method gave better accuracy than the R-K method.

Other results obtained for this problem are given in Appendix C.

PROBLEM 4 (NON-LINEAR WITH REAL EIGENVALUES)

Chemical Kinetic Problem

Gear [11] discussed the application of stiffly-stable integration formulae based on backward difference approximation of the derivative and used the following example to illustrate the method.

$$\begin{aligned} y_1' &= -0.013\, y_1 + 1000\, y_1 y_3 & y_1(0) &= 1 \\ y_2' &= 2500\, y_2 y_3 & y_2(0) &= 1 \qquad (4.7) \\ y_3' &= 0.013 y_1 - 1020 y_1 y_3 - 2500 y_2 y_3 & y_3(0) &= 0 \end{aligned}$$

This application is from chemical kinetic reaction and y represents the concentration of a very reactive species which is an intermediate in the course of the reaction and always stays small.

y_1 and y_2 are monotonically decreasing and increasing respectively while y_3 increases to a maximum and thereafter is monotonically decreasing. Hull and Watt [16] showed that y_3 is bounded above by 1.3×10^{-5} while Enright and Pryce [8] suggested the error tolerance as 2.9×10^{-4}.

The eigenvalues of system (4.7) are given by

$$\lambda_1 = 0 \ , \quad \lambda_2 = -0.00928572 \text{ and } \lambda_3 = -3500.003714$$

The general solution of (4.7) is given by

$$y_i = A_i + B_i\, e^{\lambda_2 x} + C_i\, e^{\lambda_3 x} \qquad i = 1,2,3$$

for solution $x \in [0,1]$, as $x \to 1$, $e^{\lambda_3 x} \to 0$

Thus the exact solution is given by

$$y_i = A_i + B_i\, e^{\lambda_2 x}$$

where A_i and B_i are determined using the initial conditions.

The coded program and the set of solutions obtained for $x \in [0,1]$ are given in Appendix D. The error values at $x = 1$ when compared to the exact solution are given in Table 4.4 below.

TABLE 4.4

Experimental results on non-linear stiff problem

h	method	Error $y_1(1)$	Error $y_2(1)$	Error $y_3(1)$
0.0625	SS2	-4.4×10^{-16}	4.4×10^{-16}	1.3×10^{-15}
	SS4	4.4×10^{-16}	-4.4×10^{-16}	-8.9×10^{-16}
0.1	SS2	-6.7×10^{-16}	6.7×10^{-16}	1.8×10^{-15}
	SS4	-6.7×10^{-16}	6.7×10^{-16}	1.8×10^{-15}

It will be observed from Table 4.4 that for a step length $h = 0.1$, both schemes SS2 and SS4 are identical for y_1, y_2, and y_3 and very accurate when compared to the exact solution.

PROBLEM 5

A system of stiff problem which was considered both by Gear (see Hull et al [16]) and Enright [17] is discussed in this work. While these writers discussed function calls, Jacobian evaluations and matrix inversion for the efficiency of a method, the MLMM does not require any matrix inversion, instead a Predictor-Corrector formula of a given order is used. The Jacobian evaluation is only required for the determination of the eigenvalues.

Consider the system

$$\underline{y}' = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} \underline{y} \ , \ \underline{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad (4.8)$$

$$x \varepsilon [0,10]$$

The error tolerance given by Enright is $\tau = 10^{-6}$

The eigenvalues are the non zero elements in the leading diagonal of the system, while the exact solution is

$$y_i(x) = A_i e^{\lambda_1 x} + B_i e^{\lambda_2 x}$$

where $\lambda_1 = -0.1$, $\lambda_2 = -10$ and the values of A_i and B_i determined from the initial conditions are given in Appendix E. For the purpose of comparison of results, Gear discussed variable-order, variable step stiffly-stable method, while Enright used a second derivative scheme. Denote the error of our second and fourth order formulas respectively by ER2 and ER4.

The tables of values below give solutions to problem (4.8) in the range $x \varepsilon [0,10]$

TABLE 4.5(a)

Efficiency of order 2 and 4 scheme

$h = 0.5$, $x\varepsilon$ [0,10]

method	$y_1(10)$	$y_2(10)$	$y_3(10)$	$y_4(10)$
SS2	.3678794357	.37201x10	-3.344358	-36.7879435
SS4	.3678794357	.37201x10	-3.344358	-36.7879435
True Sol.	.3678794357	.37201x10	-3.344358	-36.7879435
ER2	$3.3\text{x}10^{-16}$	$3.4\text{x}10^{-55}$	$3.1\text{x}10^{-15}$	$3.5\text{x}10^{-14}$
ER4	$1.7\text{x}10^{-16}$	$1.3\text{x}10^{-55}$	$1.3\text{x}10^{-15}$	$1.4\text{x}10^{-14}$

TABLE 4.5(b)

Local error per unit step on various methods

method	steps	max local error/ unit step	Time/sec
GEAR	422	1.06×10^{-6}	0.72
ENRIGHT	125	0.44×10^{-6}	0.44
SS2	10	3.55×10^{-14}	0.50
SS4	10	1.42×10^{-14}	0.50

CHAPTER FIVE

C O N C L U S I O N

5.1 SUMMARY

This research work shed some light into some manner by which accuracy can be improved for solution of stiff Initial Value Problems (IVPs). New algorithms have also been given to support this claim. In this thesis, solutions to stiff IVPs have been considered with methods based on a composite algorithm which allows exponential fitting. This suggests that the methods given in this thesis, basically are meant to solve stiff problems whose solutions can be expressed as exponential function.

In light of the work done, we observed that any of the schemes derived from orders 2 to 6 could be used for solutions of stiff IVPs. However, orders 2 and 4 seem to give higher accuracy than the others; though all the schemes still perform better than existing methods of the same class of problems.

For order 3 formula, two parameters each were introduced as alternative approach to the predictor and the corrector schemes, thereby leading to two values for q. One of these is fixed while the other is varied for $q \; (-\infty, 0)$ This approach inroduced by Cash [2] may give good accuracy, but the work involved may not justify better accuracy over the illustrated approach given here. Hence, the two approaches were given in chapter 2, but a good choice of the free parameters a and r led to better accuracy which made our approach preferable.

5.2 SUGGESTION FOR FURTHER WORK

The MLMM discussed in this thesis permits us, not only to evaluate the second derivative of the dependent variable y, but may be extended to third or fourth derivative formula. The approach may eventually be extended to a general l-th order derivative. Furthermore, a new calss of stiff problems having solutions in the form of trigonometric or hyperbolic function may be adapted and fitted into the sum of exponential functions. Since stiff problems have solutions with fast decaying exponents, then a solution on the left half plane may then be obtained within a small interval for which any of these oscillatory functions is decreasing.

The choice of free parameters may improve the accuracy of some methods both for 1-step MLMM and 2-step MLMM. Thus like in order 3 of our scheme, we observed that the free parameters are well chosen, so as to obtain such high accuracy. Hence, the positioning of the free parameters will affect the accuracy of the methods.

Finally, there is still a need to extend the scope of this work to derivation of methods which may be efficient when dealing with a wider calss of stiff problems. However, we conclude and affirm that the methods given in this thesis is capable of solving all systems of stiff problems for which exponential fitting is applicable.

APPENDIX A

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 1

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES12', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.25
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 2 OF PROBLEM 1
      APN=(1.0-EXP(2.*Q))/Q+2.0
      APD=1.0-EXP(2.0*Q)
      A=APN/APD
      RPN=(EXP(2.*Q)-1.0)/Q-EXP(2.*Q)-1.0
      RPD=-1.5*EXP(2.*Q)+0.5+EXP(3.*Q)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q
      RQD=1.-A*Q
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      YN1=R*Q*(RQ**1.5)+1.0+0.5*Q*(2+R)
      YD=1.-0.5*Q*(2-3*R)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,2
      H=H+0.5
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.5
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PROBLEM 1 FOR ORDER 2 h=0.25',/)
   20 FORMAT(5X,'YN2= ',E18.12/)
   10 FORMAT(5X,'A R RQ '3(F18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(25X,' Q= 'F7.4,2X/)
  100 STOP
      END
```

```
RESULT OF PROBLEM 1 FOR ORDER 2 h=0.0625

                 Q=  -.1250

A R RQ      1.041623328376      -.182344792021       .778800783071

YN2=  .778800783071E+00

H=  .125       Y=         .778800783071

YN=        1.574171795570       1.574171795570       .0000E+00

ZN=        -.016570229127       -.016570229127       .0000E+00

H=  .250       Y=         .606530659713

YN=        1.225966227079       1.225966227079       .0000E+00

ZN=        -.012904907653       -.012904907653       .0000E+00

H=  .375       Y=         .472366552741

YN=         .954783457668        .954783457668       .0000E+00

ZN=        -.010050352186       -.010050352186       .0000E+00

H=  .500       Y=         .367879441171

YN=         .743586104495        .743586104495       .0000E+00

ZN=        -.007827222153       -.007827222153       .0000E+00

H=  .625       Y=         .286504796860

YN=         .579105110162        .579105110162       .0000E+00

ZN=        -.006095846742       -.006095846742       .0000E+00

H=  .750       Y=         .223130160148

YN=         .451007770513        .451007770513       .0000E+00

ZN=        -.004747150216       -.004747150216       .0000E+00

H=  .875       Y=         .173773943450

YN=         .351245204847        .351245204847       .5551E-16

ZN=        -.003697317946       -.003697317946      -.8674E-18

H= 1.000       Y=         .135335283237

YN=         .273550040585        .273550040585       .5551E-16

ZN=        -.002879474111       -.002879474111      -.8674E-18
```

```
RESULT OF PROBLEM 1 FOR ORDER 2 h=0.25

                 Q=  -.5000

A R RQ      1.163953113739       -.601971802672        .367879441171

YN2=  .367879441171E+00

H=  .500      Y=       .367879441171

YN=       .743586104495       .743586104495       .0000E+00

ZN=      -.007827222153      -.007827222153       .0000E+00

H= 1.000      Y=       .135335283237

YN=       .273550040585       .273550040585       .0000E+00

ZN=      -.002879474111      -.002879474111       .0000E+00
```

```
      IMPLICIT real*8 (a-h,o-z)
      dimension yn(16),zn(16),exy(16),exz(16)
      open(3,file='evalte')
      a1=95.0/47.0
      c1=-1.0/47.0
      d=-2.0
      hh=0.03125
      q=d*hh
c     to evaluate y(x) and z(x) using multi-derivative linear multiste
c     method for the case k=2.
      apn=1.0+2.0*q+4.0/3.0*(q**2)+exp(2.0*q)*(2.0*(q**2)-3.0)/3.0
      apd=q**2*exp(2.0*q)-q*exp(2.0*q)+q*(q+1.0)
      a=apn/apd
      rpn=1.0+q+(q**2)/3.0-exp(2.0*q)*(1.0-q+(q**2)/3.0)
      rpd=3.0*(q**2)*(3.0*exp(2.0*q)+1.0)/4.0-q*(exp(3.0*q)-1.0)
      r=rpn/rpd
      rqn=1.0+(2.0-a)*q+(4.0/3.0-a)*(q**2)
      rqd=1.0-a*q-(2.0/3.0-a)*(q**2)
      rq=rqn/rqd
      write(3,85)apn,apd,rpn,rpd
      write(3,10)a,r,rq
      yn1=r*q*(rq**1.5)+1.0+(1.0-r)*q+(1.0/3.0-3.0*r/4.0)*(q**2)
      yd=1.0-q+(1.0/3.0+9.0*r/4.0)*(q**2)
      yn2=yn1/yd
      write(3,20)yn2
      y=1.0
      z=1.0
      h=0.0
      x=0.0
      do 30 i=1,16
      h=h+0.0625
      y=y*yn2
      yn(i)=y*a1
      zn(i)=y*c1
      write(3,40)h,y
      x=x+0.0625
      exy(i)=a1*exp(d*x)
      ery=exy(i)-yn(i)
      exz(i)=c1*exp(d*x)
      erz=exz(i)-zn(i)
      write(3,65)yn(i),exy(i),ery
      write(3,70)zn(i),exz(i),erz
   30 continue
   20 format(5x,f18.12)
   10 format(5x,3(f18.12,2x)/)
   40 format(5x,'h= ',f6.4,5x,'y= ',f15.12,/)
   65 format(5x,'yn= ',f15.12,5x,f15.12,5x,e12.5,/)
   70 format(5x,'zn= ',e18.12,5x,f15.12,5x,e12.5,//)
   85 format(5x,4f15.10,2x/)
   80 stop
      end
```

THE RESULT FOLLOWING IS FOR ORDER 4 SCHEME FOR h=0.03125

```
        .000009599      .0000095599      .0000000398      .0000003278
        1.004182438268         .121495731331         .882496902649
        .882496902588
h=  .0625     y=  .882496902588
yn=  1.783770437982      1.783770437975      -.73475E-11
zn=  -.187765292477E-01     -.018776529248      .77341E-13
h=  .1250     y=  .778800783078
yn=  1.574171886448      1.574171886435      -.12969E-10
zn=  -.165702289025E-01     -.016570228902      .13651E-12
h=  .1875     y=  .687289278799
yn=  1.389201813932      1.389201813914      -.17167E-10
zn=  -.146231756816E-01     -.014623175681      .18070E-12
h=  .2500     y=  .606530659723
yn=  1.225966297865      1.225966297844      -.20200E-10
zn=  -.129049072450E-01     -.012904907245      .21263E-12
h=  .3125     y=  .535261428530
yn=  1.081911460543      1.081911460521      -.22283E-10
zn=  -.113885406719E-01     -.011388540672      .23455E-12
h=  .3750     y=  .472366552753
yn=  .954783512804       .954783512780       -.23597E-10
zn=  -.100503518680E-01     -.010050351868      .24839E-12
h=  .4375     y=  .416862019691
yn=  .842593492692       .842593492668       -.24295E-10
zn=  -.886940439342E-02     -.008869404393      .25574E-12
h=  .5000     y=  .367879441194
yn=  .743386147442       .743386147417       -.24503E-10
zn=  -.782722190499E-02     -.007827221905      .25793E-12
h=  .5625     y=  .224652467370
```

```
yn=  .656212471925       .656212471900    -.24327E-10
zn= -.690749908703E-02  -.006907499087     .25608E-12

h=  .6250   y=  .286504796872
yn=  .579105473913       .579105473889    -.23854E-10
zn= -.609584654893E-02  -.006095846549     .25110E-12

h=  .6875   y=  .252839525816
yn=  .511058787000       .511058786977    -.23156E-10
zn= -.537956569809E-02  -.005379565698     .24375E-12

h=  .7500   y=  .223130160159
yn=  .451007796568       .451007796546    -.22293E-10
zn= -.474745006583E-02  -.004747450066     .23467E-12

h=  .8125   y=  .196911675215
yn=  .398012983515       .398012983493    -.21313E-10
zn= -.418960997829E-02  -.004189609978     .22435E-12

h=  .8750   y=  .173773943460
yn=  .351245225142       .351245225121    -.20256E-10
zn= -.369731782889E-02  -.003697317829     .21322E-12

h=  .9375   y=  .153354966854
yn=  .309972823236       .309972823217    -.19152E-10
zn= -.326287153188E-02  -.003262871532     .20160E-12

h= 1.0000   y=  .135335283236
yn=  .273550056393       .273550056375    -.18029E-10
zn= -.287947402043E-02  -.002879474020     .18978E-12
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES2', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.125
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 OF PROBLEM 1
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      YN1=R*Q*(RQ**1.5)+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,4
      H=H+0.25
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.25
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PROBLEM 1 FOR ORDER 4 h=0.125',/)
   20 FORMAT(5X,'YN2= ',E18.12/)
   10 FORMAT(5X,'A R RQ '3(F18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(25X,' Q= 'F7.4,2X/)
  100 STOP
      END
```

RESULT OF PROBLEM (1) FOR ORDER 4 h=0.0625

Q -.1250

A R RQ	1.008329615677	.124500218663	.77880078307
YN2=	.778800783071E+00		
H= .125	Y=	.778800783071	
YN+	1.574171795570	1.574171795570	.2220E-15
ZN=	-.016570229427	-.016570229427	-.3469E-17
H= .250	Y=	.606530659713	
YN+	1.225966227079	1.225966227079	.2220E-15
ZN=	-.012904907653	-.012904907653	-.3469E-17
H= .375	Y=	.472366552741	
YN+	.954783457668	.954783457668	.2220E-15
ZN=	-.010050352186	-.010050352186	-.3469E-17
H= .500	Y=	.367879441171	
YN+	.743586104495	.743586104495	.3331E-15
ZN=	-.007827222153	-.007827222153	-.3469E-17
H= .625	Y=	.286504796860	
YN+	.579105440462	.579105440462	.4441E-15
ZN=	-.006095846742	-.006095846742	-.4337E-17
H= .750	Y=	.223130160148	
YN+	.451007770513	.451007770513	.3331E-15
ZN=	-.004747450216	-.004747450216	-.3469E-17
H= .875	Y=	.173773943450	
YN+	.351245204847	.351245204847	.3331E-15
ZN=	-.003697317946	-.003697317946	-.3469E-17
H= 1.000	Y=	.135335283237	
YN+	.273550040585	.273550040585	.2776E-15
ZN=	-.002879474111	-.002879474111	-.3036E-17

RESULT FOR ORDER 4 h=0.125

1.016636987186	.130583414558	.606530659713

YN2 .606530659713

H= .250	Y=	.606530659713	
YN+	1.225966227079	1.225966227079	.0000E+00
ZN=	-.012904907653	-.012904907653	.0000E+00
H= .500	Y=	.367879441171	
YN+	.743586104495	.743586104495	.0000E+00
ZN=	-.007827222153	-.007827222153	.0000E+00
H= .750	Y=	.223130160148	
YN+	.451007770513	.451007770513	-.1110E-15
ZN=	-.004747450216	-.004747450216	.8674E-18
H= 1.000	Y=	.135335283237	
YN+	.273550040585	.273550040585	.0000E+00
ZN=	-.002879474111	-.002879474111	.0000E+00

```fortran
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES3', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=0.0625
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 5 OF PROBLEM 1
      APN=-1.+(1./6.)*(Q**2)-2.*Q*EXP(Q)+(1.-(Q**2)/6.)*EXP(2.*Q
     1)
      APD=Q*(1.+0.5*Q)-2.*Q*EXP(Q)+Q*(1.0-0.5*Q)*EXP(2.*Q)
      A=APN/APD
      RPN=1.0+(7./15.)*Q+(Q**2)/15.+16.*Q/15.*EXP(Q)-(1.-7.*Q/1
     15.+(Q**2)/15.)*EXP(2.*Q)
      RPD=-0.5*Q*(7.+3.*Q)+9.*Q*EXP(Q)-4.5*Q*EXP(2.*Q)*(1.-Q)-Q*
     1EXP(3.0*Q)
      R=RPN/RPD
C     USING EXPONENTIAL FUNCTION IN EVALUATING R(q)
      RQN=1.+A*Q-(Q**2)*(1.-3.*A)/6.+2*Q*(1.-A)*EXP(Q)
      RQD=1.-A*Q-(Q**2)*(1.-3.*A)/6.0
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      R1Q=RQ**0.5
      R2Q=RQ**1.5
      YN1=1.+Q*(7./15.+3.5*R)+(1./15.+1.5*R)*(Q**2)+(16./15.-9.*R
     1)*Q*R1Q+R*Q*R2Q
      YD=1.-(7./15.+4.5*R)*Q+(1./15.+4.5*R)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,8
      H=H+0.1250
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.125
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'EXPO. RES PRB 1, CASE K=2 ORDER 5 h=0.0625',/)
   20 FORMAT(5X,'YN2= ',E18.12/)
   10 FORMAT(5X,'A R RQ ',3(F18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN=',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(25X,' Q= ',F7.4,2X/)
  100 STOP
      END
```

EXPO. RES PRB 1, CASE K=2 ORDER 5 h=0.0625

Q= -.1250

A R RQ .466626999630 -.000183739779 .778800783071

YN2= .778800783071E+00

H= .125	Y=	.778800783071	
YN=	1.574171795570	1.574171795570	.0000E+00
ZN=	-.016570229427	-.016570229427	.0000E+00
H= .250	Y=	.606530659713	
YN=	1.225966227079	1.225966227079	.0000E+00
ZN=	-.012904907653	-.012904907653	.0000E+00
H= .375	Y=	.472366552741	
YN=	.954783457668	.954783457668	.0000E+00
ZN=	-.010050352186	-.010050352186	.0000E+00
H= .500	Y=	.367879441171	
YN=	.743586104495	.743586104495	.0000E+00
ZN=	-.007827222153	-.007827222153	.0000E+00
H= .625	Y=	.286504796860	
YN=	.579105440462	.579105440462	.0000E+00
ZN=	-.006095846742	-.006095846742	.0000E+00
H= .750	Y=	.223130160148	
YN=	.451007770513	.451007770513	.0000E+00
ZN=	-.004747450216	-.004747450216	.0000E+00
H= .875	Y=	.173773943450	
YN=	.351245204847	.351245204847	.5551E-16
ZN=	-.003697317946	-.003697317946	-.8674E-18
H= 1.000	Y=	.135335283237	
YN=	.273550040585	.273550040585	.5551E-16
ZN=	-.002879474111	-.002879474111	-.8674E-18

PADE-RESULT OF PROBL 1 FOR ORDER 5 h=0.0625

Q = -.1250

A R RQ .466626999630E+00 -.183739778859E-03 .778818903546E+00

YN2= .778799489296E+00

H= .125	Y=	.778799489296	
YN=	1.574169180493	1.574171795570	.2615E-05
ZN=	-.016570201900	-.016570229427	-.2753E-07
H= .250	Y=	.606528644528	
YN=	1.225962153834	1.225966227079	.4073E-05
ZN=	-.012904864777	-.012904907653	-.4288E-07
H= .375	Y=	.472364198602	
YN=	.954778699303	.954783457668	.4758E-05
ZN=	-.010050302098	-.010050352186	-.5009E-07
H= .500	Y=	.367876996633	
YN=	.743581163408	.743586104495	.4941E-05
ZN=	-.007827170141	-.007827222153	-.5201E-07
H= .625	Y=	.286502417102	
YN=	.579100630313	.579105440462	.4810E-05
ZN=	-.006095796109	-.006095846742	-.5063E-07
H= .750	Y=	.223127936121	
YN=	.451003275139	.451007770513	.4495E-05
ZN=	-.004747402896	-.004747450216	-.4732E-07
H= .875	Y=	.173771922699	
YN=	.351241120349	.351245204847	.4084E-05
ZN=	-.003697274951	-.003697317946	-.4299E-07
H= 1.000	Y=	.135333484652	
YN=	.273546405148	.273550040585	.3635E-05
ZN=	-.002879435844	-.002879474111	-.3827E-07

EXPO. RESULT FOR CASE K=2 ORDER 5 h=0./125

Q= -.2500

A R RQ	.466508184129	-.000382298798	.606530659713
YN2=	.606530659713E+00		
H= .250	Y=	.606530659713	
YN=	1.225966227079	1.225966227079	.0000E+00
ZN=	-.012904907653	-.012904907653	.0000E+00
H= .500	Y=	.367879441171	
YN=	.743586104495	.743586104495	.0000E+00
ZN=	-.007827222153	-.007827222153	.0000E+00
H= .750	Y=	.223130160148	
YN=	.451007770513	.451007770513	-.1110E-15
ZN=	-.004747450216	-.004747450216	.8674E-18
H= 1.000	Y=	.135335283237	
YN=	.273550040585	.273550040585	.0000E+00
ZN=	-.002879474111	-.002879474111	.0000E+00

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES10', STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=0.05
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 6 ON PROBLEM 1
      APN=3.0+14.0*Q/5.+(2./3.)*(Q**2)+(4.*Q/15.)*EXP(Q)*(12.0+7.
     10*Q)+EXP(2.0*Q)*(4.*(Q**2)-45.0)/15.0
      APD=Q*(3.0*Q)+4.*(Q**2)*EXP(Q)-Q*EXP(2.*Q)*(3.-Q)
      A=APN/APD
      RPN=1.+7.*Q/15.+(Q**2)/15.+16.*Q*EXP(Q)/15.-EXP(2.*Q)*(1.0-
     17.*Q/15.+(Q**2)/15.0)
      RPD=Q*(10.+3.*Q)+9.*Q*EXP(Q)*(1.+2.*Q)-9*Q*EXP(2.*Q)*(2.-Q)
     1-Q*EXP(3.0*Q)
      R=RPN/RPD
C     EVALUATE R(Q) USING PADE FOR Y(N+1)/Y(N)
      RQN=1.0+Q*(1.5-A)+(Q**2)*(41./45.-7.*A/6.)+(Q**3)*(.2-.5*A)
      RQD=1.0-Q*(A+0.5)-(Q**2)*(4.0/45.-5.0*A/6.0)+(Q**3)*(2./45
     1.0-A/6.0)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      S=RQ**0.5
      T=RQ**1.5
      YN1=1.0+(7./15.-10.0*R)*Q+(1./15.-3.*R)*(Q**2)+R*Q*T+S*((1
     16./15.-9.*R)*Q-18.*R*(Q**2))
      YD=1.-(7/15.+18.*R)*Q+(1./15.+9.*R)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.1
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.1
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'PADE-RESULT OF PROBLEM 1 FOR ORDER 6 h=.05',/)
   20 FORMAT(5X,'YN2= 'E18.12)
   10 FORMAT(5X,'A R RQ   ',3(E12.6,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(25X,'Q = ',F6.4,2X/)
  100 STOP
      END
```

PADE-RESULT OF PROBLEM 1 FOR ORDER 6 h=.0625

Q = -.1250

A R RQ .129658E-06 .438503E-02 .778819E+00

YN2= .778799546412E+00
H= .125 Y= .778799546412

YN=	1.574169295939	1.574171795570	.2500E-05
ZN=	-.016570203115	-.016570229427	-.2631E-07

H= .250 Y= .606528733492

YN=	1.225962333653	1.225966227079	.3893E-05
ZN=	-.012904866670	-.012904907653	-.4098E-07

H= .375 Y= .472364302529

YN=	.954778909367	.954783457068	.4548E-05
ZN=	-.010050304309	-.010050352186	-.4788E-07

H= .500 Y= .367877104551

YN=	.743581381539	.743586104495	.4723E-05
ZN=	-.007827172437	-.007827222153	-.4972E-07

H= .625 Y= .286502522160

YN=	.579100842663	.579105440462	.4598E-05
ZN=	-.006095798344	-.006095846742	-.4840E-07

H= .750 Y= .223128034304

YN=	.451003473593	.451007770513	.4297E-05
ZN=	-.004747404985	-.004747450216	-.4523E-07

H= .875 Y= .173772011908

YN=	.351241300664	.351245204847	.3904E-05
ZN=	-.003697276849	-.003697317946	-.4110E-07

H= 1.000 Y= .135333564053

YN=	.273546565639	.273550040585	.3475E-05
ZN=	-.002879437533	-.002879474111	-.3658E-07

PADE-RESULT OF PROBLEM 1 FOR ORDER 6 h=.05

Q = -.1000

A R RQ .441857E-07 .435442E-02 .818738E+00

YN2= .818730334674E+00

H= .100	Y=	.818730334674	
YN=	1.654880463704	1.654881304481	.8408E-06
ZN=	-.017419794355	-.017419803205	-.8850E-08
H= .200	Y=	.670319360916	
YN=	1.354900835894	1.354902212634	.1377E-05
ZN=	-.014262114062	-.014262128554	-.1449E-07
H= .300	Y=	.548810794702	
YN=	1.109298414823	1.109300105591	.1691E-05
ZN=	-.011676825419	-.011676843217	-.1780E-07
H= .400	Y=	.449328045619	
YN=	.908216262422	.908218108134	.1846E-05
ZN=	-.009560171183	-.009560190612	-.1943E-07
H= .500	Y=	.367878501168	
YN=	.743584204489	.743586093415	.1889E-05
ZN=	-.007827202153	-.007827222036	-.1988E-07
H= .600	Y=	.301193288381	
YN=	.608794944600	.608796800426	.1856E-05
ZN=	-.006408367838	-.006408387373	-.1954E-07
H= .700	Y=	.246596081798	
YN=	.498438888741	.498440661399	.1773E-05
ZN=	-.005246725145	-.005246743804	-.1866E-07

H= .800	Y= .201895692580		
YN=	.408087038193	.408088696855	.1659E-05
ZN=	-.004295653034	-.004295670493	-.1746E-07
H= .900	Y= .165298127955		
YN=	.334113237356	.334114765103	.1528E-05
ZN=	-.003516981446	-.003516997527	-.1608E-07
H= 1.000	Y= .135334591622		
YN=	.273548642640	.273550032432	.1390E-05
ZN=	-.002879459396	-.002879474026	-.1463E-07

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES9',STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.005
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 7 FOR PROBLEM 1
C     REPLACE Y(n+1)/Y(n) BY EXP Q
      RQN=15.+7.*Q+Q**2+16.*Q*EXP(Q)
      RQD=15.0-7.*Q+Q**2
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,10)Q,RQ
      R1Q=RQ**0.5
      R2Q=RQ**1.5
      YN1=1.0+1049.*Q/1890.+37.*(Q**2)/315.+2.*R2Q/945.+R1Q*(34.*
     1Q/35.+4.*(Q**2)/105.)
      YD=1.-33.*Q/70.+8*(Q**2)/105.
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,25
      H=H+0.01
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.01
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'EXP FN RESULT FOR ORDER 7 h=0.005 OF PROBLEM 1',
     1/)
   20 FORMAT(5X,' YN2 =',E18.12/)
   10 FORMAT(5X,' Q=',F10.4,5X,'RQ= 'F18.12,/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(5X,4F15.10,2X/)
  100 STOP
      END
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RES7',STATUS = 'NEW')
      A1=95.0/47.0
      C1=-1.0/47.0
      D=-2.0
      HH=.005
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 7 ON PROBLEM 1
      RQN=30.+31.*Q+11.*(Q**2)-Q**3
      RQD=30.0-29.*Q+9.*(Q**2)-Q**3
C     EVALUATE R(q) USING PADE APPROXIMATION FOR Y(n+1)/Y(n)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,10)Q,RQ
      R1Q=RQ**0.5
      R2Q=RQ**1.5
      YN1=1.0+1049.*Q/1890.+37.*(Q**2)/315.+2.*R2Q/945.+R1Q*(34.*
     1Q/35.+4.*(Q**2)/105.)
      YD=1.-33.*Q/70.+8*(Q**2)/105.
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,20
      H=H+0.01
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.02
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'PADE APPROX.RESULT FOR ORDER 7 h=0.005 OF
     1PROBLEM 1',/)
   20 FORMAT(5X,'YN2= ',F18.12)
   10 FORMAT(5X,'Q=',F10.4,5X,'RQ=',F18.12,/)
   40 FORMAT(5X,'H=',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN=',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN=',F18.12,5X,F15.12,5X,E10.4,/)
  100 STOP
      END
```

EXP FN RESULT FOR ORDER 7 h=0.0625 OF PROBLEM 1

Q= -.1250 RQ= .778800783156

YN2 = .780361932486E+00

H= .125	Y= .780361932486		
YN=	1.577327310344	1.574171795570	-.3156E-02
ZN=	-.016603445372	-.016570229427	.3322E-04
H= .250	Y= .608964745673		
YN=	1.230886188063	1.225966227079	-.4920E-02
ZN=	-.012956696716	-.012904907653	.5179E-04
H= .375	Y= .475212905749		
YN=	.960536724387	.954783457668	-.5753E-02
ZN=	-.010110912888	-.010050352186	.6056E-04
H= .500	Y= .370838061473		
YN=	.749566294466	.743586104495	-.5980E-02
ZN=	-.007890171521	-.007827222153	.6295E-04
H= .625	Y= .289387906290		
YN=	.584933002076	.579105440462	-.5828E-02
ZN=	-.006157189496	-.006095846742	.6134E-04
H= .750	Y= .225827305791		
YN=	.456459447875	.451007770513	-.5452E-02
ZN=	-.004804836293	-.004747450216	.5739E-04
H= .875	Y= .176227032755		
YN=	.356203576845	.351245204847	-.4958E-02
ZN=	-.003749511335	-.003697317946	.5219E-04
H= 1.000	Y= .137520867837		
YN=	.277967711585	.273550040585	-.4418E-02
ZN=	-.002925975911	-.002879474111	.4650E-04

PADE APPROX.RESULT FOR ORDER 7 h=0.0625 OF PROBLEM 1

Q= -.1250RQ= .778818902192

YN2= .780360810273
H= .125 Y= .780360810273

YN=	1.577325042042	1.574171795570	-.3153E-02
ZN=	-.016603421495	-.016570229427	.3319E-04

H= .250 Y= .608962994210

YN=	1.230882647872	1.225966227079	-.4916E-02
ZN=	-.012956659451	-.012904907653	.5175E-04

H= .375 Y= .475210855588

YN=	.960532580445	.954783457668	-.5749E-02
ZN=	-.010110869268	-.010050352186	.6052E-04

H= .500 Y= .370835928318

YN=	.749561982770	.743586104495	-.5976E-02
ZN=	-.007890126134	-.007827222153	.6290E-04

H= .625 Y= .289385825500

YN=	.584928796224	.579105440462	-.5823E-02
ZN=	-.006157145223	-.006095846742	.6130E-04

H= .750 Y= .225825357269

YN=	.456455509374	.451007770513	-.5448E-02
ZN=	-.004804794836	-.004747450216	.5734E-04

H= .875 Y= .176225258779

YN=	.356199991149	.351245204847	-.4955E-02
ZN=	-.003749473591	-.003697317946	.5216E-04

H= 1.000 Y= .137519285731

YN=	.277964513712	.273550040585	-.4414E-02
ZN=	-.002925942250	-.002879474111	.4647E-04

EXP FN RESULT FOR ORDER 7 h=0.005 OF PROBLEM 1

Q= -.0100 RQ= .980198673745

YN2 = .982263325691E+00

H= .010	Y= .982263325691		
YN=	1.985425871077	1.981252638421	-.4173E-02
ZN=	-.020899219696	-.020855290931	.4393E-04
H= .020	Y= .964841240997		
YN=	1.950211019037	1.942021208534	-.8190E-02
ZN=	-.020528537042	-.020442328511	.8621E-04
H= .030	Y= .947728166146		
YN=	1.915620761359	1.903566612989	-.1205E-01
ZN=	-.020164429067	-.020037543295	.1269E-03
H= .040	Y= .930918620329		
YN=	1.881644019815	1.865873469437	-.1577E-01
ZN=	-.019806779156	-.019640773362	.1660E-03
H= .050	Y= .914407219952		
YN=	1.848269912669	1.828926700118	-.1934E-01
ZN=	-.019455472765	-.019251860001	.2036E-03
H= .060	Y= .898188676906		
YN=	1.815487751193	1.792711525832	-.2278E-01
ZN=	-.019110397381	-.018870647640	.2397E-03
H= .070	Y= .882257796876		
YN=	1.783287036238	1.757213460028	-.2607E-01
ZN=	-.018771442487	-.018496983790	.2745E-03
H= .080	Y= .866609477676		
YN=	1.751657454877	1.722418303006	-.2924E-01
ZN=	-.018438499525	-.018130718979	.3078E-03
H= .090	Y= .851238707617		
YN=	1.720588877098	1.688312136241	-.3228E-01
ZN=	-.018111461864	-.017771706697	.3398E-03

H= .100	Y=	.836140563901	
YN=	1.690071352566	1.654881316811	-.3519E-01
ZN=	-.017790224764	-.017419803335	.3704E-03
H= .110	Y=	.821310211042	
YN=	1.660095107426	1.622112471943	-.3798E-01
ZN=	-.017474685341	-.017074868126	.3998E-03
H= .120	Y=	.806742899322	
YN=	1.630650541183	1.589992493664	-.4066E-01
ZN=	-.017164742539	-.016736763091	.4280E-03
H= .130	Y=	.792433963266	
YN=	1.601728223623	1.558508533554	-.4322E-01
ZN=	-.016860297091	-.016405352985	.4549E-03
H= .140	Y=	.778378820148	
YN=	1.573318891788	1.527647997610	-.4567E-01
ZN=	-.016561251493	-.016080505238	.4807E-03
H= .150	Y=	.764572968526	
YN=	1.545413447020	1.497398541206	-.4801E-01
ZN=	-.016267509969	-.015762089907	.5054E-03
H= .160	Y=	.751011986798	
YN=	1.518002952038	1.467748064158	-.5025E-01
ZN=	-.015978978443	-.015449979623	.5290E-03
H= .170	Y=	.737691531786	
YN=	1.491078628077	1.438684705879	-.5239E-01
ZN=	-.015695564506	-.015144049536	.5515E-03
H= .180	Y=	.724607337346	
YN=	-.010973489411	-.010565644927	.4078E-03
ZN=	-.015417177390	-.014844177270	.5730E-03
H= .190	Y=	.711755213001	
YN=	-.010778856203	-.010356431145	.4224E-03
ZN=	-.010565420555	-.010151360073	.4141E-03

H= .200 Y= .699131042600

YN= -.010587675141 -.015113334187 -.4526E-02

ZN= -.010378025131 -.014814070126 -.4436E-02

H= .210 Y= .686730782998

YN= -.010399884995 -.015111800056 -.4712E-02

ZN= -.010193953480 -.014812566373 -.4619E-02

H= .220 Y= .674550462762

YN= -.010215425622 -.015110266081 -.4895E-02

ZN= -.010013146647 -.014811062772 -.4798E-02

H= .230 Y= .662586180899

YN= -.010034237945 -.015108732261 -.5074E-02

ZN= -.009835546726 -.014809559324 -.4974E-02

H= .240 Y= .650834105607

YN= -.009856263935 -.015107198597 -.5251E-02

ZN= -.009661096837 -.014808056029 -.5147E-02

H= .250 Y= .639290473047

YN= -.009681446591 -.015105665089 -.5424E-02

ZN= -.009489741109 -.014806552886 -.5317E-02

PADE APPROX.RESULT FOR ORDER 7 h=0.005 OF PROBLEM 1

Q= -.0100 RQ= .980198674621

YN2= .982263325689

H= .010	Y= .982263325689		
YN=	1.985425871074	1.942021208534	-.4340E-01
ZN=	-.020899219696	-.020442328511	.4569E-03
H= .020	Y= .964841240994		
YN=	1.950211019031	1.865873469437	-.8434E-01
ZN=	-.020528537042	-.019640773362	.8878E-03
H= .030	Y= .947728166141		
YN=	1.915620761350	1.792711525832	-.1229E+00
ZN=	-.020164429067	-.018870647640	.1294E-02
H= .040	Y= .930918620324		
YN=	1.881644019803	1.722418303006	-.1592E+00
ZN=	-.019806779156	-.018130718979	.1676E-02
H= .050	Y= .914407219945		
YN=	1.848269912655	1.654881316811	-.1934E+00
ZN=	-.019455472765	-.017419803335	.2036E-02
H= .060	Y= .898188676898		
YN=	1.815487751176	1.589992493664	-.2255E+00
ZN=	-.019110397381	-.016736763091	.2374E-02
H= .070	Y= .882257796866		
YN=	1.783287036219	1.527647997610	-.2556E+00
ZN=	-.018771442487	-.016080505238	.2691E-02
H= .080	Y= .866609477665		
YN=	1.751657454855	1.467748064158	-.2839E+00
ZN=	-.018438499525	-.015449979623	.2989E-02
H= .090	Y= .851238707605		
YN=	1.720588877074	1.410196840640	-.3104E+00
ZN=	-.018111461864	-.014844177270	.3267E-02
H= .100	Y= .836140563888		
YN=	1.690071352539	1.354902232824	-.3352E+00
ZN=	-.017790224764	-.014262128767	.3528E-02

H= .110	Y=	.821310211028	
YN=	1.660095107397	1.301775757545	-.3583E+00
ZN=	-.017474685341	-.013702902711	.3772E-02
H= .120	Y=	.806742899307	
YN=	1.630650541153	1.250732401112	-.3799E+00
ZN=	-.017164742538	-.013165604222	.3999E-02
H= .130	Y=	.792433963250	
YN=	1.601728223590	1.201690483269	-.4000E+00
ZN=	-.016860297090	-.012649373508	.4211E-02
H= .140	Y=	.778378820131	
YN=	1.573318891754	1.154571526487	-.4187E+00
ZN=	-.016561251492	-.012153384489	.4408E-02
H= .150	Y=	.764572968608	
YN=	1.545413446984	1.109300130386	-.4361E+00
ZN=	-.016267509968	-.011676843478	.4591E-02
H= .160	Y=	.751011986779	
YN=	1.518002952000	1.065803851078	-.4522E+00
ZN=	-.015978978442	-.011218987906	.4760E-02
H= .170	Y=	.737691531766	
YN=	1.491078628038	1.024013085239	-.4671E+00
ZN=	-.015695564506	-.010779085108	.4916E-02
H= .180	Y=	.724607337325	
YN=	-.007810604159	-.005246744078	.2564E-02
ZN=	-.015417177390	-.010356431145	.5061E-02
H= .190	Y=	.711755212980	
YN=	-.007672070017	-.005041016304	.2631E-02
ZN=	-.007371243855	-.004843355232	.2528E-02
H= .200	Y=	.699131042579	
YN=	-.007535993009	-.010758222548	-.3222E-02
ZN=	-.007240502503	-.010336386618	-.3096E-02

APPENDIX B

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 2

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='EB2', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-1.0
      P=D(1)-D(2)
      V(1)=-9909.0/P
      V(2)=-1000.0/P
      V(3)=9.0/P
      V(4)=-0.1/P
      W(1)=9909.0/P
      W(2)=1000.0/P
      W(3)=-9.0/P
      W(4)=0.1/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.1
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,5
      H=H+0.2
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
```

```
      WRITE(3,40)H,Y
      X=X+0.2
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
      WRITE(3,80)Y3N(I),EXY3(I),ER3
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 4 h=0.1 ON PROBLEM 2',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'A R RQ',3(E15.6,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'Y1= ',F18.12,5X,F18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2= ',F18.12,5X,F18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3= ',F18.12,5X,F18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4= ',F18.12,5X,F18.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT FOR ORDER 4 h=0.05 ON PROBLEM 2

V(I)=	-.1101E+05	-.1111E+04	.1000E+02	-.1111E+00
W(I)=	.1101E+05	.1111E+04	-.1000E+02	.1111E+00
A R RQ	.100033E+01	.118728E+00	.990050E+00	
A R RQ	.100333E+01	.120898E+00	.904837E+00	
YP(I)=	.990049833454	.904837416688		

H= .100 Y= .990049833454

Y1=	-938.188710152108	-938.188710152108	.0000E+00
Y2=	-94.680463230609	-94.680463230609	.0000E+00
Y3=	.852124169075	.852124169075	.0000E+00
Y4=	-.009468046464	-.009468046464	.0000E+00

H= .200 Y= .980198672723

Y1=	-1777.761825094169	-1777.761825094169	.0000E+00
Y2=	-179.408802613197	-179.408802613197	.0000E+00
Y3=	1.614679223519	1.614679223519	.0000E+00
Y4=	-.017940880529	-.017940880529	.0000E+00

H= .300 Y= .970445532681

Y1=	-2528.196745758465	-2528.196745758464	.1364E-11
Y2=	-255.141461878945	-255.141461878945	.1137E-12
Y3=	2.296273156911	2.296273156911	-.1332E-14
Y4=	-.025514146568	-.025514146568	.1388E-16

H= .400 Y= .960789438007

Y1=	-3198.068054888938	-3198.068054888937	.1364E-11
Y2=	-322.743773830754	-322.743773830754	.1137E-12
Y3=	2.904693964477	2.904693964477	-.1332E-14
Y4=	-.032274377864	-.032274377864	.1388E-16

H= .500 Y= .951229423083

Y1=	-3795.133440748525	-3795.133440748522	.2728E-11
Y2=	-382.998631622618	-382.998631622618	.2274E-12
Y3=	3.446987684604	3.446987684604	-.2220E-14
Y4=	-.038299863733	-.038299863733	.2776E-16

H= .600	Y=	.941764531900	
Y1=	-4326.411444013024	-4326.411444013021	.3638E-11
Y2=	-436.614334848423	-436.614334848423	.3979E-12
Y3=	3.929529013636	3.929529013636	-.3109E-14
Y4=	-.043661434135	-.043661434135	.3469E-16
H= .700	Y=	.932393817961	
Y1=	-4798.251805979109	-4798.251805979105	.4547E-11
Y2=	-484.231688967515	-484.231688967515	.3979E-12
Y3=	4.358085200708	4.358085200708	-.4441E-14
Y4=	-.048423169618	-.048423169618	.4163E-16
H= .800	Y=	.923116344186	
Y1=	-5216.399122165437	-5216.399122165433	.3638E-11
Y2=	-526.430429121550	-526.430429121549	.4547E-12
Y3=	4.737873862094	4.737873862094	-.3553E-14
Y4=	-.052643043697	-.052643043697	.4857E-16
H= .900	Y=	.913931182820	
Y1=	-5586.050438383845	-5586.050438383841	.4547E-11
Y2=	-563.735032635366	-563.735032635366	.4547E-12
Y3=	5.073615293718	5.073615293718	-.3553E-14
Y4=	-.056373504104	-.056373504104	.4163E-16
H= 1.000	Y=	.904837415339	
Y1=	-5911.907365731702	-5911.907365731698	.4547E-11
Y2=	-596.619978376395	-596.619978376395	.4547E-12
Y3=	5.369579805388	5.369579805388	-.4441E-14
Y4=	-.059661998727	-.059661998727	.4857E-16

RESULT FOR ORDER 4 h=0.1 ON PROBLEM 2

V(I)=	-.1101E+05	-.1111E+04	.1000E+02	-.1111E+00
W(I)=	.1101E+05	.1111E+04	-.1000E+02	.1111E+00

A R RQ	.100067E+01	.118993E+00	.980199E+00
A R RQ	.100666E+01	.123295E+00	.818731E+00

YP(I)= .980198672723 .818730750638

H= .200 Y= .980198672723

Y1=	-1777.761825094168	-1777.761825094169	-.1137E-11
Y2=	-179.408802613197	-179.408802613197	-.1137E-12
Y3=	1.614679223519	1.614679223519	.1110E-14
Y4=	-.017940880529	-.017940880529	-.1041E-16

H= .400 Y= .960789438007

Y1=	-3198.068054888935	-3198.068054888937	-.2274E-11
Y2=	-322.743773830753	-322.743773830754	-.2842E-12
Y3=	2.904693964477	2.904693964477	.2220E-14
Y4=	-.032274377864	-.032274377864	-.2082E-16

H= .600 Y= .941764531900

Y1=	-4326.411444013019	-4326.411444013021	-.1819E-11
Y2=	-436.614334848423	-436.614334848423	-.2274E-12
Y3=	3.929529013636	3.929529013636	.2220E-14
Y4=	-.043661434135	-.043661434135	-.2082E-16

H= .800 Y= .923116344186

Y1=	-5216.399122165430	-5216.399122165433	-.3638E-11
Y2=	-526.430429121549	-526.430429121549	-.3411E-12
Y3=	4.737873862094	4.737873862094	.3553E-14
Y4=	-.052643043697	-.052643043697	-.3469E-16

H= 1.000 Y= .904837415339

Y1=	-5911.907365731694	-5911.907365731698	-.3638E-11
Y2=	-596.619978376395	-596.619978376395	-.3411E-12
Y3=	5.369579805388	5.369579805388	.2665E-14
Y4=	-.059661998727	-.059661998727	-.3469E-16

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SB2', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-1.0
      P=D(1)-D(2)
      V(1)=-9909.0/P
      V(2)=-1000.0/P
      V(3)=9.0/P
      V(4)=-0.1/P
      W(1)=9909.0/P
      W(2)=1000.0/P
      W(3)=-9.0/P
      W(4)=0.1/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.005
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 7
      RQN=15.+7.*Q+Q**2+16.*Q*EXP(Q)
      RQD=15.-7.*Q+Q**2
      RQ=RQN/RQD
      WRITE(3,10)RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T1=RQ**0.5
      T=RQ**1.5
      YN1=1.+1049.*Q/1890.+37.*(Q**2)/315.+2.*T/945.+T1*(34.*Q/35
     1.+4.*(Q**2)/105.)
      YD=1.-33.*Q/70.+8.*(Q**2)/105.
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,20
      H=H+0.01
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+0.01
```

```
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
      WRITE(3,80)Y3N(I),EXY3(I),ER3
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 7 h=0.005 ON PROBLEM 2',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'RQ',E15.6,2X/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y=',F18.12,/)
   65 FORMAT(5X,'Y1=',F18.12,5X,F18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2=',F18.12,5X,F18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3=',F18.12,5X,F18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4=',F18.12,5X,F18.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT FOR ORDER 7 h=0.05 ON PROBLEM 2

V(I)=	-.1101E+05	-.1111E+04	.1000E+02	-.1111E+00
W(I)=	.1101E+05	.1111E+04	-.1000E+02	.1111E+00

RQ .990050E+00

RQ .904837E+00

YP(I)= .992140220779 .906706869786

H= .100	Y= .992140220779		
Y1=	-940.621195996117	-938.188710152108	.2432E+01
Y2=	-94.925945705532	-94.680463230609	.2455E+00
Y3=	.854333511350	.852124169075	-.2209E-02
Y4=	-.009492594712	-.009468046464	.2455E-04
H= .200	Y= .984342217688		
Y1=	-1786.095821341269	-1777.761825094169	.8334E+01
Y2=	-180.249855822108	-179.408802613197	.8411E+00
Y3=	1.622248702399	1.614679223519	-.7569E-02
Y4=	-.018024985851	-.017940880529	.8411E-04
H= .300	Y= .976605505180		
Y1=	-2545.358505377342	-2528.196745758464	.1716E+02
Y2=	-256.873398463754	-255.141461878945	.1732E+01
Y3=	2.311860586174	2.296273156911	-.1559E-01
Y4=	-.025687340229	-.025514146568	.1732E-03
H= .400	Y= .968929601523		
Y1=	-3226.509881192059	-3198.068054888937	.2844E+02
Y2=	-325.614076212742	-322.743773830754	.2870E+01
Y3=	2.930526685915	2.904693964477	-.2583E-01
Y4=	-.032561408106	-.032274377864	.2870E-03
H= .500	Y= .961314028775		
Y1=	-3836.894395329803	-3795.133440748522	.4176E+02
Y2=	-387.213078547765	-382.998631622618	.4214E+01
Y3=	3.484917706930	3.446987684604	-.3793E-01
Y4=	-.038721308432	-.038299863733	.4214E-03

H= .600	Y= .953758312747		
Y1=	-4383.170858362740	-4326.411444013021	.5676E+02
Y2=	-442.342401691668	-436.614334848423	.5728E+01
Y3=	3.981081615225	3.929529013636	-.5155E-01
Y4=	-.044234240828	-.043661434135	.5728E-03
H= .700	Y= .946261982979		
Y1=	-4871.376413550332	-4798.251805979105	.7312E+02
Y2=	-491.611304223467	-484.231688967515	.7380E+01
Y3=	4.424501738011	4.358085200708	-.6642E-01
Y4=	-.049161131155	-.048423169618	.7380E-03
H= .800	Y= .938824572708		
Y1=	-5306.984537634598	-5216.399122165433	.9059E+02
Y2=	-535.572160423312	-526.430429121549	.9142E+01
Y3=	4.820149443810	4.737873862094	-.8228E-01
Y4=	-.053557216840	-.052643043697	.9142E-03
H= .900	Y= .931445618840		
Y1=	-5694.957630532069	-5586.050438383841	.1089E+03
Y2=	-574.725767537801	-563.735032635366	.1099E+02
Y3=	5.172531907840	5.073615293718	-.9892E-01
Y4=	-.057472577610	-.056373504104	.1099E-02
H= 1.000	Y= .924124661920		
Y1=	-6039.794698741433	-5911.907365731698	.1279E+03
Y2=	-609.526157911135	-596.619978376395	.1291E+02
Y3=	5.485735421200	5.369579805388	-.1162E+00
Y4=	-.060952616699	-.059661998727	.1291E-02

APPENDIX C

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 3

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RS1', STATUS = 'NEW')
      A1=1.0
      C1=-1.0
      D=-1.0
      HH=0.05
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT
C     ISTEP METHOD FOR THE CASE K=2 ORDER 3 PROBLEM 3
      APN=(EXP(2.*Q)-1.)/Q-2.-Q*(EXP(2.*Q)+1.)
      APD=-1.-Q+(1.-Q)*EXP(2.*Q)
      A=APN/APD
      RPN=(EXP(2.*Q)-1.)/Q-1.-EXP(2.*Q)+Q*(EXP(2.*Q)-1.)/3.
      RPD=EXP(3.*Q)+EXP(2.*Q)*(4.+5.*Q)/4.-(8.+15.*Q)/4.
      R=RPN/RPD
      RQN=1.0+Q*(2.0-A)+(Q**2)*(1.0-A)
      RQD=1.0-A*Q-(1.0-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      T=RQ**1.5
      YN1=4.0*R*Q*T-R*Q*(8.+15.*Q)+4.*(1.+Q+(Q**2)/3.)
      YD=4.*(1.-Q+(Q**2)/3.)-R*Q*(4.+5.*Q)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=-1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.1
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.1
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
 30   CONTINUE
 99   FORMAT(5X,'RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 3 USING H
     1=0.05',/)
 20   FORMAT(5X,'YN2= ',E18.12/)
 10   FORMAT(5X,'A R RQ  ',3(E12.6,2X)/)
 40   FORMAT(5X,'H= ',F6.4,5X,'Y= ',F18.12,/)
 65   FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
 70   FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
 85   FORMAT(25X,'Q= ',F7.4,2X/)
100   STOP
      END
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 3 USING h = 0.05

```
                     Q=   -.0500
A R RQ   -.190000E+02   -.255324E-05    .904547E+00
YN2=  .904837416637E+00
H=   .1000       Y=        .904837416637
YN=       .904837416637        .904837416688        .5040E 10
ZN=      -.904837416637       -.904837416688       -.5040E-10
H=   .2000       Y=        .818730750547
YN=       .818730750547        .818730750638        .9121E-10
ZN=      -.818730750547       -.818730750638       -.9121E-10
H=   .3000       Y=        .740818217246
YN=       .740818217246        .740818217370        .1238E 09
ZN=      -.740818217246       -.740818217370       -.1238E-09
H=   .4000       Y=        .670320041891
YN=       .670320041891        .670320042040        .1494E-09
ZN=      -.670320041891       -.670320042040       -.1494E 09
H=   .5000       Y=        .606530655025
YN=       .606530655025        .606530655194        .1689E 09
ZN=      -.606530655025       -.606530655194       -.1689E-09
H=   .6000       Y=        .548811631004
YN=       .548811631004        .548811631187        .1834E-09
ZN=      -.548811631004       -.548811631187       -.1834E-09
H=   .7000       Y=        .496585298418
YN=       .496585298418        .496585298612        .1936E 09
ZN=      -.496585298418       -.496585298612       -.1936E 09
H=   .8000       Y=        .449328958561
YN=       .449328958561        .449328958761        .2002E-09
ZN=      -.449328958561       -.449328958761       -.2002E-09
```

H= .9000	Y=	.406569654084	
YN=	.406569654084	.406569654288	.2038E 09
ZN=	-.406569654084	-.406569654288	-.2038E-09
H= 1.0000	Y=	.367879435485	
YN=	.367879435485	.367879435690	.2049E 09
ZN=	-.367879435485	-.367879435690	-.2049E-09

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='SOL1', STATUS = 'NEW')
      A1=1.0
      C1=-1.0
      D=-1.0
      HH=0.1
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 PROBLEM 3
      APN=1.+2.*Q+4.*(Q**2)/3.+EXP(2.*Q)*(2.*(Q**2)-3.)/3.
      APD=Q*EXP(2.*Q)*(Q-1.)+Q*(1.+Q)
      A=APN/APD
      RPN=1.+Q+(Q**2)/3.-EXP(2.*Q)*(1.-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2*Q)+1)/4.0-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.0+Q*(2.0-A)+(Q**2)*(4.0/3.-A)
      RQD=1.0-Q*A-(Q**2)*(2./3.-A)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      T=RQ**1.5
      YN1=R*Q*T+1.+Q*(1.-R)+(Q**2)*(1./3.-(3./4.)*R)
      YD=1.-Q+(Q**2)*(1./3.+(9./4.)*R)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=-1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.2
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.2
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
30    CONTINUE
99    FORMAT(5X,'RESULT OF PROBLEM  3 FOR CASE K=2 ORDER 4 USING H
     1=0.1',/)
20    FORMAT(5X,'YN2= ',E18.12)
10    FORMAT(5X,'A R RQ '3(E12.6,2X)/)
40    FORMAT(5X,'H= ',F6.4,5X,'Y= ',F18.12,/)
65    FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
70    FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
85    FORMAT(25X,'Q= ',F6.2,2X/)
100   STOP
      END
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING h =0.0027

Q= .00

A R RQ .100018E+01 .118102E+00 .994615E+00

YN2= .994614553652E+00

H= .0054	Y= .994614553652		
YN=	.994614553652	.994614553652	.0000E+00
ZN=	-.994614553652	-.994614553652	.0000E+00
H= .0108	Y= .989258110337		
YN=	.989258110337	.989258110337	-.1110E-15
ZN=	-.989258110337	-.989258110337	.1110E-15
H= .0162	Y= .983930513859		
YN=	.983930513859	.983930513859	-.1110E-15
ZN=	-.983930513859	-.983930513859	.1110E-15
H= .0216	Y= .978631608867		
YN=	.978631608867	.978631608867	-.1110E-15
ZN=	-.978631608867	-.978631608867	.1110E-15
H= .0270	Y= .973361240843		
YN=	.973361240843	.973361240843	-.2220E-15
ZN=	-.973361240843	-.973361240843	.2220E-15
H= .0324	Y= .968119256103		
YN=	.968119256103	.968119256103	-.2220E-15
ZN=	-.968119256103	-.968119256103	.2220E-15
H= .0378	Y= .962905501791		
YN=	.962905501791	.962905501791	-.2220E-15
ZN=	-.962905501791	-.962905501791	.2220E-15
H= .0432	Y= .957719825873		
YN=	.957719825873	.957719825873	-.3331E-15
ZN=	-.957719825873	-.957719825873	.3331E-15

H= .0486	Y=	.952562077134	
YN=	.952562077134	.952562077134	-.3331E 15
ZN=	-.952562077134	-.952562077134	.3331E-15
H= .0540	Y=	.947432105175	
YN=	.947432105175	.947432105175	-.4441E 15
ZN=	-.947432105175	-.947432105175	.4441E 15
H= .0594	Y=	.942329760404	
YN=	.942329760404	.942329760404	-.4441E 15
ZN=	-.942329760404	-.942329760404	.4441E 15
H= .0648	Y=	.937254894037	
YN=	.937254894037	.937254894037	-.3331E 15
ZN=	-.937254894037	-.937254894037	.3331E 15
H= .0702	Y=	.932207358091	
YN=	.932207358091	.932207358091	-.3331E 15
ZN=	-.932207358091	-.932207358091	.3331E 15
H= .0756	Y=	.927187005379	
YN=	.927187005379	.927187005379	-.4441E 15
ZN=	-.927187005379	-.927187005379	.4441E 15
H= .0810	Y=	.922193689507	
YN=	.922193689507	.922193689507	-.4441E 15
ZN=	-.922193689507	-.922193689507	.4441E 15
H= .0864	Y=	.917227264870	
YN=	.917227264870	.917227264870	-.5551E 15
ZN=	-.917227264830	-.917223264870	.5551E 15
YN=	.912287586659	.912287586646	-.1276E 10
ZN=	-.912287586659	-.912287586646	.1276E 10

H= .0972	Y=	.907374510808	
YN=	-.827786502649	-.827786502637	.1226E-10
ZN=	-.907374510808	-.907374510794	.1344E-10
H= .1026	Y=	.902487894063	
YN=	-.823328502852	-.823328502839	.1287E-10
ZN=	-.818894511373	-.818894511360	.1280E-10
H= .1080	Y= 360	.4441E-15	
H= .1080	Y=	.897627593916	
YN=	-.818894511360	-.835069259634	-.1617E-01
ZN=	-.814484398905	-.830572038939	-.1609E-01

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING h = 0.05

Q= -.05

A R RQ .100333E+01 .120898E+00 .904837E+00

YN2= .904837416688E+00

H= .1000	Y= .904837416688		
YN=	.904837416688	.904837416688	.0000E+00
ZN=	-.904837416688	-.904837416688	.0000E+00
H= .2000	Y= .818730750638		
YN=	.818730750638	.818730750638	.0000E+00
ZN=	-.818730750638	-.818730750638	.0000E+00
H= .3000	Y= .740818217370		
YN=	.740818217370	.740818217370	.0000E+00
ZN=	-.740818217370	-.740818217370	.0000E+00
H= .4000	Y= .670320042040		
YN=	.670320042040	.670320042040	.0000E+00
ZN=	-.670320042040	-.670320042040	.0000E+00
H= .5000	Y= .606530655194		
YN=	.606530655194	.606530655194	.1110E-15
ZN=	-.606530655194	-.606530655194	-.1110E-15
H= .6000	Y= .548811631187		
YN=	.548811631187	.548811631187	.1110E-15
ZN=	-.548811631187	-.548811631187	-.1110E-15
H= .7000	Y= .496585298612		
YN=	.496585298612	.496585298612	.1665E-15
ZN=	-.496585298612	-.496585298612	-.1665E-15
H= .8000	Y= .449328958761		
YN=	.449328958761	.449328958761	.1665E-15
ZN=	-.449328958761	-.449328958761	-.1665E-15

```
H=  .9000       Y=        .406569654288

YN=        .406569654288         .406569654288       .1665E-15

ZN=       -.406569654288        -.406569654288      -.1665E-15

H= 1.0000       Y=        .367879435690

YN=        .367879435690         .367879435690       .1665E-15

ZN=       -.367879435690        -.367879435690      -.1665E-15
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 4 USING h = 0.1

Q= -.10

A R RQ .100666E+01 .123295E+00 .818731E+00

YN2= .818730750638E+00

H= .2000	Y=	.818730750638	
YN=	.818730750638	.818730750638	-.1110E-15
ZN=	-.818730750638	-.818730750638	.1110E-15
H= .4000	Y=	.670320042040	
YN=	.670320042040	.670320042040	-.2220E-15
ZN=	-.670320042040	-.670320042040	.2220E-15
H= .6000	Y=	.548811631187	
YN=	.548811631187	.548811631187	-.2220E-15
ZN=	-.548811631187	-.548811631187	.2220E-15
H= .8000	Y=	.449328958761	
YN=	.449328958761	.449328958761	-.2220E-15
ZN=	-.449328958761	-.449328958761	.2220E-15
H= 1.0000	Y=	.367879435690	
YN=	.367879435690	.367879435690	-.2220E-15
ZN=	-.367879435690	-.367879435690	.2220E-15

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16)
      OPEN(UNIT = 3, FILE='RS5', STATUS = 'NEW')
      A1=1.0
      C1=-1.0
      D=-1.0
      HH=0.05
      Q=D*HH
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 6 PROBLEM 3
      APN=3.0+14.0*Q/5.+(2./3.)*(Q**2)+(4.*Q/15.)*EXP(Q)*(12.0+7.
     10*Q)+EXP(2.0*Q)*(4.*(Q**2)-45.0)/15.0
      APD=Q*(3.0*Q)+4.*(Q**2)*EXP(Q)-Q*EXP(2.*Q)*(3.-Q)
      A=APN/APD
      RPN=1.+7.*Q/15.+(Q**2)/15.+16.*Q*EXP(Q)/15.-EXP(2.*Q)*(1.0-
     17.*Q/15.+(Q**2)/15.0)
      RPD=Q*(10.+3.*Q)+9.*Q*EXP(Q)*(1.+2.*Q)-9*Q*EXP(2.*Q)*(2.-Q)
     1-Q*EXP(3.0*Q)
      R=RPN/RPD
C     EVALUATING R(q) BY FITTING PADE APPROX.INTO ITS FORM
      RQN=1.0+Q*(1.5-A)+(Q**2)*(41./45.-7.*A/6.)+(Q**3)*(.2-.5*A)
      RQD=1.0-Q*(A+0.5)-(Q**2)*(4.0/45.-5.0*A/6.0)+(Q**3)*(2./45
     1.0-A/6.0)
      RQ=RQN/RQD
      WRITE(3,99)
      WRITE(3,10)A,R,RQ
      S=RQ**0.5
      T=RQ**1.5
      YN1=1.0+(7./15.-10.0*R)*Q+(1./15.-3.*R)*(Q**2)+R*Q*T+S*((1
     16./15.-9.*R)*Q-18.*R*(Q**2))
      YD=1.-(7/15.+18.*R)*Q+(1./15.+9.*R)*(Q**2)
      YN2=YN1/YD
      WRITE(3,20)YN2
      Y=1.0
      Z=-1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+0.1
      Y=Y*YN2
      YN(I)=Y*A1
      ZN(I)=Y*C1
      WRITE(3,40)H,Y
      X=X+0.1
      EXY(I)=A1*EXP(D*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=C1*EXP(D*X)
      ERZ=EXZ(I)-ZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 6 USING h
     1=0.05',/)
   20 FORMAT(5X,'YN2',F18.12/)
   10 FORMAT(5X,'A R RQ',3(E12.6,2X)/)
   40 FORMAT(5X,'H= ',F6.4,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 6 USING h =0.0027

```
A = -.100743E-13     R = -.453438E+03     RQ = .994615E+00

YN2        .994614553653

H=  .0054      Y=        .994614553653

YN=        .994614553653         .994614553652        -.8182E-12

ZN=       -.994614553653        -.994614553652         .8182E-12

H=  .0108      Y=        .989258110338

YN=        .989258110338         .989258110337        -.1628E-11

ZN=       -.989258110338        -.989258110337         .1628E 11

H=  .0162      Y=        .983930513861

YN=        .983930513861         .983930513859        -.2428E-11

ZN=       -.983930513861        -.983930513859         .2428E-11

H=  .0216      Y=        .978631608870

YN=        .978631608870         .978631608867        -.3220E-11

ZN=       -.978631608870        -.978631608867         .3220E-11

H=  .0270      Y=        .973361240847

YN=        .973361240847         .973361240843        -.4004E-11

ZN=       -.973361240847        -.973361240843         .4004E-11

H=  .0324      Y=        .968119256108

YN=        .968119256108         .968119256103        -.4779E 11

ZN=       -.968119256108        -.968119256103         .4779E-11

H=  .0378      Y=        .962905501796

YN=        .962905501796         .962905501791        -.5545E-11

ZN=       -.962905501796        -.962905501791         .5545E-11

H=  .0432      Y=        .957719825879

YN=        .957719825879         .957719825873        -.6303E-11

ZN=       -.957719825879        -.957719825873         .6303E-11
```

H= .0486	Y=	.952562077141	
YN=	.952562077141	.952562077134	-.7053E-11
ZN=	-.952562077141	-.952562077134	.7053E-11
H= .0540	Y=	.947432105183	
YN=	.947432105183	.947432105175	-.7794E-11
ZN=	-.947432105183	-.947432105175	.7794E-11
H= .0594	Y=	.942329760413	
YN=	.942329760413	.942329760404	-.8528E-11
ZN=	-.942329760413	-.942329760404	.8528E-11
H= .0648	Y=	.937254894047	
YN=	.937254894047	.937254894037	-.9253E 11
ZN=	-.937254894047	-.937254894037	.9253E-11
H= .0702	Y=	.932207358101	
YN=	.932207358101	.932207358091	-.9970E-11
ZN=	-.932207358101	-.932207358091	.9970E-11
H= .0756	Y=	.927187005390	
YN=	.927187005390	.927187005379	-.1068E-10
ZN=	-.927187005390	-.927187005379	.1068E-10
H= .0810	Y=	.922193689518	
YN=	.922193689518	.922193689507	-.1138E-10
ZN=	-.922193689518	-.922193689507	.1138E 10
H= .0864	Y=	.917227264882	
YN=	.917227264882	.917227264870	-.1207E 10
ZN=	-.917227264882	-.917227264870	.1207E-10
H= .0918	Y=	.912287586659	
YN=	.912287586659	.912287586646	-.1276E-10
ZN=	-.912287586659	-.912287586646	.1276E-10

H= .0972	Y=	.907374510808	
YN=	-.827786502649	-.827786502637	.1226E-10
ZN=	-.907374510808	-.907374510794	.1344E-10
H= .1026	Y=	.902487894063	
YN=	-.823328502852	-.823328502839	.1287E-10
ZN=	-.818894511373	-.818894511360	.1280E-10

RESULT OF PROBLEM 3 FOR CASE K=2 ORDER 6 USING h = 0.05

A = .149485E-08   R = .429431E-02   RQ = .904838E+00

YN2 .904837402983

H= .1000	Y= .904837402983		
YN=	.904837402983	.904837416688	.1370E-07
ZN=	-.904837402983	-.904837416688	-.1370E-07
H= .2000	Y= .818730725836		
YN=	.818730725836	.818730750638	.2480E-07
ZN=	-.818730725836	-.818730750638	-.2480E-07
H= .3000	Y= .740818183708		
YN=	.740818183708	.740818217370	.3366E-07
ZN=	-.740818183708	-.740818217370	-.3366E-07
H= .4000	Y= .670320001429		
YN=	.670320001429	.670320042040	.4061E-07
ZN=	-.670320001429	-.670320042040	-.4061E-07
H= .5000	Y= .606530609260		
YN=	.606530609260	.606530655194	.4593E-07
ZN=	-.606530609260	-.606530655194	-.4593E-07
H= .6000	Y= .548811581312		
YN=	.548811581312	.548811631187	.4987E-07
ZN=	-.548811581312	-.548811631187	-.4987E-07
H= .7000	Y= .496585245962		
YN=	.496585245962	.496585298612	.5265E-07
ZN=	-.496585245962	-.496585298612	-.5265E-07
H= .8000	Y= .449328904315		
YN=	.449328904315	.449328958761	.5445E-07
ZN=	-.449328904315	-.449328958761	-.5445E-07

H= .9000	Y=	.406569598866	
YN=	.406569598866	.406569654288	.5542E-07
ZN=	-.406569598866	-.406569654288	-.5542E-07
H= 1.0000	Y=	.367879379969	
YN=	.367879379969	.367879435690	.5572E-07
ZN=	-.367879379969	-.367879435690	-.5572E-07

APPENDIX D

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 4

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16),EXZZ(16),D(6),Q1(
     16),C(5),ZZN(16),V(4),YP(4)
      OPEN(UNIT = 3, FILE='RST2', STATUS = 'NEW')
      D(1)=0.0
      D(2)=-0.00928572
      D(3)=-3500.003714
      V(1)=(D(2)+0.013)/D(2)
      V(2)=1.0+32.5/(D(2)*D(3))
      V(3)=(0.013*(D(3)-D(2))+45.5002)/(D(2)*D(3))
      B1=(13.00002+0.013*D(3))/(D(2)*(D(2)-D(3)))
      B2=-32.5/(D(2)*(D(3)-D(2)))
      B3=(0.013*D(3)-45.5002)/(D(2)*(D(3)-D(2)))
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),B1,B2,B3
      HH=0.1
      DO 31 K=1,3
      Q1(K)=D(K)*HH
      Q=Q1(K)
      IF(Q.EQ.0.0) GO TO 44
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 2 ON PROBLEM 4
      APN=(1.0-EXP(2.*Q))/Q+2.0
      APD=1.-EXP(2.0*Q)
      A=APN/APD
      RPN=(EXP(3.*Q)-1.)/Q-EXP(2.*Q)-1.0
      RPD=EXP(3.*Q)-3.*EXP(2.*Q)/2.+0.5
      R=RPN/RPD
      RQN=1.+(2.-A)*Q
      RQD=1.-A*Q
      RQ=RQN/RQD
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+0.5*Q*(2.+R)
      YD=1.-0.5*Q*(2.-3.*R)
      YP(K)=YN1/YD
   44 C(K)=V(K)
   31 CONTINUE
      WRITE(3,50)C(1),C(2),C(3)
      WRITE(3,20)YP(1),YP(2),YP(3)
      Y=1.0
      Z=1.0
      ZZ=0.0
      H=0.0
      X=0.0
      DO 33 I=1,5
      H=H+0.2
      Y=Y*YP(2)
      YN(I)=V(1)+Y*B1
      ZN(I)=V(2)+Y*B2
      ZZN(I)=V(3)+Y*B3
      WRITE(3,40)H,Y
      X=X+0.2
      EXY(I)=V(1)+B1*EXP(D(2)*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=V(2)+B2*EXP(D(2)*X)
      ERZ=EXZ(I)-ZN(I)
      EXZZ(I)=V(3)+B3*EXP(D(2)*X)
      ERZZ=EXZZ(I)-ZZN(I)
```

```
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
      WRITE(3,80)ZZN(I),EXZZ(I),ERZZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 2 h=0.1 ON PROBLEM 4 ',/)
   90 FORMAT(5X,'V(I) B',6(F8.4,2X)/)
   50 FORMAT(5X,'C(I)=',3(F6.4,2X)/)
   20 FORMAT(5X,'YP(I) ='3(E10.4,2X)/)
   10 FORMAT(5X,'A R RQ',3(E18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   80 FORMAT(5X,'ZZ= ',F18.12,5X,F15.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT FOR ORDER 2 h=0.0625 ON PROBLEM 4

V(I) B -.4000 2.0000 .0000 1.0000 -1.0000 -2.8000

A R RQ .100019345253E+01 -.444616506177E+00 .998839958361E+00

A R RQ .199542857621E+01 -.199085715242E+01 -.229793622073E-16

C(I)=-.4000 2.0000 .0000
YP(I) = .0000E+00 .9988E+00 -.1152E-16

H= .125	Y=	.998839958361	
YN=	.598842660537	.598842660537	.0000E+00
ZN=	1.001157389718	1.001157389718	.0000E+00
ZZ=	-2.796753900738	-2.796753900738	.0000E+00
H= .250	Y=	.997681262419	
YN=	.597683962437	.597683962437	-.1110E-15
ZN=	1.002316086801	1.002316086801	.2220E-15
ZZ=	-2.793509540102	-2.793509540102	.4441E-15
H= .375	Y=	.996523910612	
YN=	.596526608475	.596526608475	-.1110E-15
ZN=	1.003473439747	1.003473439747	.2220E-15
ZZ=	-2.790268943061	-2.790268943061	.4441E-15
H= .500	Y=	.995367901381	
YN=	.595370597092	.595370597092	-.2220E-15
ZN=	1.004629450116	1.004629450116	.2220E-15
ZZ=	-2.787032105246	-2.787032105246	.8882E-15
H= .625	Y=	.994213233169	
YN=	.594215926731	.594215926731	-.3331E-15
ZN=	1.005784119464	1.005784119464	.2220E-15
ZZ=	-2.783799022299	-2.783799022299	.8882E-15
H= .750	Y=	.993059904421	
YN=	.593062595835	.593062595835	-.3331E-15
ZN=	1.006937449348	1.006937449348	.4441E-15
ZZ=	-2.780569689862	-2.780569689862	.8882E-15

H= .875   Y= .991907913581

YN=	.591910602851	.591910602851	-.4441E-15
ZN=	1.008089441321	1.008089441321	.4441E-15
ZZ=	-2.777344103585	-2.777344103585	.1332E-14

H= 1.000   Y= .990757259100

YN=	.590759946227	.590759946227	-.4441E-15
ZN=	1.009240096936	1.009240096936	.4441E-15
ZZ=	-2.774122259123	-2.774122259123	.1332E-14

RESULT FOR ORDER 2 h=0.1 ON PROBLEM 4

V(I) B -.4000 2.0000 .0000 1.0000 -1.0000 -2.8000

A R RQ .100030952399E+01 -.444719629669E+00 .998144579384E+00

A R RQ .199714286017E+01 -.199428572035E+01 -.180203502993E-16

C(I)=-.4000 2.0000 .0000

YP(I) = .0000E+00 .9981E+00 -.9023E-17

H= .200	Y=	.998144579384	
YN=	.598147280265	.598147280265	-.1110E-15
ZN=	1.001852769379	1.001852769379	.2220E-15
ZZ=	-2.794806832403	-2.794806832403	.4441E-15
H= .400	Y=	.996292601354	
YN=	.596295298787	.596295298787	-.2220E-15
ZN=	1.003704749233	1.003704749233	.2220E-15
ZZ=	-2.789621274745	-2.789621274745	.4441E-15
H= .600	Y=	.994444059522	
YN=	.594446753514	.594446753514	-.3331E-15
ZN=	1.005553292884	1.005553292884	.4441E-15
ZZ=	-2.784445338477	-2.784445338477	.8882E-15
H= .800	Y=	.992598947513	
YN=	.592601638069	.592601638069	-.5551E-15
ZN=	1.007398406709	1.007398406709	.6661E-15
ZZ=	-2.779279005748	-2.779279005748	.1332E-14
H= 1.000	Y=	.990757258963	
YN=	.590759946090	.590759946090	-.6661E-15
ZN=	1.009240097073	1.009240097073	.6661E-15
ZZ=	-2.774122258739	-2.774122258739	.1776E-14

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION YN(16),ZN(16), EXY(16),EXZ(16),EXZZ(16),D(6),Q1(
     16),C(6),ZZN(16),V(4),YP(4)
      OPEN(UNIT = 3, FILE='TAL', STATUS = 'NEW')
      D(1)=0.0
      D(2)=-0.00928572
      D(3)=-3500.003714
      V(1)=(D(2)+0.013)/D(2)
      V(2)=1.0+32.5/(D(2)*D(3))
      V(3)=(0.013*(D(3)-D(2))+45.5002)/(D(2)*D(3))
      B1=(13.00002+0.013*D(3))/(D(2)*(D(2)-D(3)))
      B2=-32.5/(D(2)*(D(3)-D(2)))
      B3=(0.013*D(3)-45.5002)/(D(2)*(D(3)-D(2)))
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),B1,B2,B3
      HH=0.0625
      DO 31 K=1,3
      Q1(K)=D(K)*HH
      Q=Q1(K)
      IF(Q.EQ.0.0)GO TO 44
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 OF PROBLEM 4
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q-(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,85)Q
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YP(K)=YN1/YD
   44 C(K)=V(K)
   31 CONTINUE
      WRITE(3,50)C(1),C(2),C(3)
      WRITE(3,20)YP(1),YP(2),YP(3)
      Y=1.0
      Z=1.0
      ZZ=0.0
      H=0.0
      X=0.0
      DO 30 I=1,8
      H=H+0.125
      Y=Y*YP(2)
      YN(I)=V(1)+Y*B1
      ZN(I)=V(2)+Y*B2
      ZZN(I)=V(3)+Y*B3
      WRITE(3,40)H,Y
```

```
      X=X+0.125
      EXY(I)=V(1)+B1*EXP(D(2)*X)
      ERY=EXY(I)-YN(I)
      EXZ(I)=V(2)+B2*EXP(D(2)*X)
      ERZ=EXZ(I)-ZN(I)
      EXZZ(I)=V(3)+B3*EXP(D(2)*X)
      ERZZ=EXZZ(I)-ZZN(I)
      WRITE(3,65)YN(I),EXY(I),ERY
      WRITE(3,70)ZN(I),EXZ(I),ERZ
      WRITE(3,80)ZZN(I),EXZZ(I),ERZZ
   30 CONTINUE
   99 FORMAT(5X,'RESULT OF PRB 4 FOR ORDER 4 h=0.0625',/)
   90 FORMAT(5X,'V(I) B',6(F8.4,2X)/)
   50 FORMAT(5X,'C(I)=',3(F6.4,2X)/)
   20 FORMAT(5X,'YP(K) ',3(F18.12,2X)/)
   10 FORMAT(5X,'A R RQ',3(E18.12,2X)/)
   40 FORMAT(5X,'H= ',F5.3,5X,'Y= ',F18.12,/)
   65 FORMAT(5X,'YN= ',F18.12,5X,F15.12,5X,E10.4,/)
   70 FORMAT(5X,'ZN= ',F18.12,5X,F15.12,5X,E10.4,/)
   80 FORMAT(5X,'ZZ= ',F18.12,5X,F15.12,5X,E10.4,/)
   85 FORMAT(5X,'Q = ',F12.6,2X/)
  100 STOP
      END
```

RESULT OF PROBLEM 4 FOR ORDER 4 h=0.0625

V(I) B -.4000 2.0000 .0000 1.0000 -1.0000 -2.8000

Q = -.000580

A R RQ .999709043900E+00 -.893235908462E+00 .998839958361E+00

Q = -218.750229

A R RQ .133029271544E+01 .441065472874E+00 -.706390904169E-16

C(I)=-.4000 2.0000 .0000

YP(K) .0000E+00 .9988E+00 -.2812E-17

H= .125 Y= .998839958361

YN=	.598842660537	.598842660537	.1110E-15
ZN=	1.001157389718	1.001157389718	-.2220E-15
ZZ=	-2.796753900738	-2.796753900738	-.4441E-15

H= .250 Y= .997681262419

YN=	.597683962437	.597683962437	.1110E-15
ZN=	1.002316086801	1.002316086801	.0000E+00
ZZ=	-2.793509540102	-2.793509540102	-.4441E-15

H= .375 Y= .996523910612

YN=	.596526608475	.596526608475	.2220E-15
ZN=	1.003473439747	1.003473439747	-.2220E-15
ZZ=	-2.790268943061	-2.790268943061	-.4441E-15

H= .500 Y= .995367901381

YN=	.595370597092	.595370597092	.2220E-15
ZN=	1.004629450116	1.004629450116	-.2220E-15
ZZ=	-2.787032105246	-2.787032105246	-.4441E-15

H= .625 Y= .994213233169

YN=	.594215926731	.594215926731	.2220E-15
ZN=	1.005784119464	1.005784119464	-.2220E-15
ZZ=	-2.783799022299	-2.783799022299	-.4441E-15

H= .750	Y=	.993059904421	
YN=	.593062595835	.593062595835	.3331E-15
ZN=	1.006937449348	1.006937449348	-.2220E-15
ZZ=	-2.780569689862	-2.780569689862	-.8882E-15
H= .875	Y=	.991907913581	
YN=	.591910602851	.591910602851	.3331E-15
ZN=	1.008089441321	1.008089441321	-.2220E-15
ZZ=	-2.777344103585	-2.777344103585	-.8882E-15
H= 1.000	Y=	.990757259100	
YN=	.590759946227	.590759946227	.4441E-15
ZN=	1.009240096936	1.009240096936	-.4441E-15
ZZ=	-2.774122259123	-2.774122259123	-.8882E-15

RESULT OF PROBLEM 4 FOR ORDER 4 h=0.1

V(I) B -.4000 2.0000 .0000 1.0000 -1.0000 -2.8000

Q = -.000929

A R RQ .100005272217E+01 .100631094613E+00 .998144579384E+00

Q = -350.000371

A R RQ .133143130233E+01 .442330877229E+00 -.273108666792E-15

C(I)=-.4000 2.0000 .0000

YP(K) .0000E+00 .9981E+00 -.2556E-16

H= .200	Y=	.998144579384	
YN=	.598147280265	.598147280265	-.1110E-15
ZN=	1.001852769379	1.001852769379	.2220E-15
ZZ=	-2.794806832403	-2.794806832403	.4441E-15
H= .400	Y=	.996292601354	
YN=	.596295298787	.596295298787	-.2220E-15
ZN=	1.003704749233	1.003704749233	.2220E-15
ZZ=	-2.789621274745	-2.789621274745	.4441E-15
H= .600	Y=	.994444059522	
YN=	.594446753514	.594446753514	-.3331E-15
ZN=	1.005553292884	1.005553292884	.4441E-15
ZZ=	-2.784445338477	-2.784445338477	.8882E-15
H= .800	Y=	.992598947513	
YN=	.592601638069	.592601638069	-.5551E-15
ZN=	1.007398406709	1.007398406709	.6661E-15
ZZ=	-2.779279005748	-2.779279005748	.1332E-14
H= 1.000	Y=	.990757258963	
YN=	.590759946090	.590759946090	-.6661E-15
ZN=	1.009240097073	1.009240097073	.6661E-15
ZZ=	-2.774122258739	-2.774122258739	.1776E-14

```
RESULT OF PROBLEM 4 FOR ORDER 4 h=0.2
V(I) B  -.4000    2.0000     .0000     1.0000    -1.0000    -2.8000
Q =      -.001857
A R RQ .100012433237E+01    .119108667004E+00    .996292601354E+00
Q =  -700.000743
A R RQ .133238163464E+01    .443386951321E+00    .192585872647E-15
C(I)=-.4000  2.0000   .0000
YP(K)  .0000E+00   .9963E+00   .1044E-16
H=  .400      Y=       .996292601354
YN=       .596295298787       .596295298787       .0000E+00
ZN=      1.003704749233      1.003704749233       .0000E+00
ZZ=     -2.789621274745     -2.789621274745       .0000E+00
H=  .800      Y=       .992598947513
YN=       .592601638069       .592601638069       .0000E+00
ZN=      1.007398406709      1.007398406709       .0000E+00
ZZ=     -2.779279005748     -2.779279005748       .0000E+00
H= 1.200      Y=       .988918987519
YN=       .588921671224       .588921671224       .1110E-15
ZN=      1.011078370326      1.011078370326      -.2220E-15
ZZ=     -2.768975079665     -2.768975079665       .0000E+00
H= 1.600      Y=       .985252670604
YN=       .585255347482       .585255347482       .0000E+00
ZN=      1.014744690850      1.014744690850       .0000E+00
ZZ=     -2.758709354343     -2.758709354343       .0000E+00
H= 2.000      Y=       .981599946187
YN=       .581602616265       .581602616265       .0000E+00
ZN=      1.018397418862      1.018397418862       .0000E+00
ZZ=     -2.748481688158     -2.748481688158       .0000E+00
```

APPENDIX E

PROGRAMS AND NUMERICAL VALUES FOR PROBLEM 5

Each program is for specific order k - scheme and the step length for each program is varied to obtain various solutions thereafter.

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16), Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SLTN1', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-10.0
      P=D(1)-D(2)
      V(1)=(-0.1-D(2))/P
      V(2)=(-10.0-D(2))/P
      V(3)=(-100.0-D(2))/P
      V(4)=(-1000.0-D(2))/P
      W(1)=(0.1+D(1))/P
      W(2)=(10.0+D(1))/P
      W(3)=(100.0+D(1))/P
      W(4)=(1000.0+D(1))/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.5
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT
C     ISTEP METHOD FOR THE CASE K=2 ORDER [illegible] ON PROBLEM [illegible]
      APN=(1.0-EXP(2.*Q))/Q+2.
      APD=1.0-EXP(2.*Q)
      A=APN/APD
      BPN=(-1.0+EXP(2.*Q))/Q-EXP(2.*Q)-1.0
      BPD=EXP(2.*Q)-1.5*EXP(2.*Q)+0.5
      B=BPN/BPD
      PQ=(1.0+(2.0-A)*Q)/(1.0-A*Q)
      WRITE(3,11)Q
      WRITE(3,10)A,B,PQ
      T=PQ**1.5
      YN1=B*Q*T+1.0+0.5*Q*(2.+B)
      YD=1.-0.5*Q*(2.0-3.*B)
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+1.0
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+1.0
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
      WRITE(3,80)Y3N(I),EXY3(I),ER3
```

```
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 2 h=0.5 ON PROBLEM 5',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'A R RQ',3(E15.6,2X)/)
   11 FORMAT(5X,' Q = ',2(F10.4,2X)/)
   40 FORMAT(5X,'H= ',F6.3,5X,'Y= ',E18.12,/)
   65 FORMAT(5X,'Y1= ',E18.12,5X,E18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2= ',E18.12,5X,E18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3= ',E18.12,5X,E18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4= ',E18.12,5X,E18.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT FOR ORDER 2 h=0.5 ON PROBLEM 5

V(I)=	.1000E+01	.0000E+00	-.9091E+01	-.1000E+03
W(I)=	.0000E+00	.1000E+01	.1009E+02	.1010E+03

Q = -.0500

A R RQ .101666E+01 -.459400E+00 .904837E+00

Q = -5.0000

A R RQ .180009E+01 -.160033E+01 .453999E-04

YP(I)= .904837416688 .000045399930

H= 1.000 Y= .904837416688E+00

Y1=	.904837416688E+00	.904837416688E+00	-.1110E-15
Y2=	.453999297625E-04	.453999297625E-04	.4005E-17
Y3=	-.822533657183E+01	-.822533657183E+01	.0000E+00
Y4=	-.904791562895E+02	-.904791562895E+02	.1421E-13

H= 2.000 Y= .818730750638E+00

Y1=	.818730750638E+00	.818730750638E+00	-.1110E-15
Y2=	.206115362244E-08	.206115362244E-08	.3635E 21
Y3=	-.744300680430E+01	-.744300680430E+01	.8882E-15
Y4=	-.818730748679E+02	-.818730748679E+02	.1421E-13

H= 3.000 Y= .740818217370E+00

Y1=	.740818217370E+00	.740818217370E+00	-.2220E-15
Y2=	.935762296884E-13	.935762296884E-13	.2475E-25
Y3=	-.673471106801E+01	-.673471106801E+01	.1776E-14
Y4=	-.740818217481E+02	-.740818217481E+02	.2842E-13

H= 4.000 Y= .670320042040E+00

Y1=	.670320042040E+00	.670320042040E+00	-.2220E-15
Y2=	.424835425529E-17	.424835425529E-17	.1498E-29
Y3=	-.609381856492E+01	-.609381856492E+01	.1776E-14
Y4=	-.670320042141E+02	-.670320042141E+02	.1421E-13

H= 5.000 Y= .606530655194E+00

Y1=	.606530655194E+00	.606530655194E+00	-.2220E-15
Y2=	.192874984796E-21	.192874984796E-21	.8504E-34
Y3=	-.551391504804E+01	-.551391504804E+01	.1776E-14
Y4=	-.606530655285E+02	-.606530655285E+02	.2132E-13

H= 6.000 Y= .548811631187E+00

Y1=	.548811631187E+00	.548811631187E+00	-.3331E-15
Y2=	.875651076269E-26	.875651076270E-26	.4632E-38
Y3=	-.498919664791E+01	-.498919664791E+01	.3553E-14
Y4=	-.548811631270E+02	-.548811631270E+02	.2842E-13

H= 7.000 Y= .496585298612E+00

Y1=	.496585298612E+00	.496585298612E+00	-.3331E-15
Y2=	.397544973591E-30	.397544973591E-30	.2453E-42
Y3=	-.451441180624E+01	-.451441180624E+01	.2665E-14
Y4=	-.496585298686E+02	-.496585298686E+02	.3553E-13

H= 8.000 Y= .449328958761E+00

Y1=	.449328958761E+00	.449328958761E+00	-.3331E-15
Y2=	.180485138784E-34	.180485138785E-34	.1273E-46
Y3=	-.408480871662E+01	-.408480871662E+01	.2665E-14
Y4=	-.449328958828E+02	-.449328958828E+02	.3553E-13

H= 9.000 Y= .406569654288E+00

Y1=	.406569654288E+00	.406569654288E+00	-.3331E-15
Y2=	.819401262398E-39	.819401262399E-39	.6501E-51
Y3=	-.369608776681E+01	-.369608776681E+01	.3109E-14
Y4=	-.406569654349E+02	-.406569654349E+02	.3553E-13

H= 10.000   Y= .367879435690E+00

Y1=	.367879435690E+00	.367879435690E+00	-.3331E-15
Y2=	.372007597602E-43	.372007597602E-43	.3280E-55
Y3=	-.334435850677E+01	-.334435850677E+01	.3109E-14
Y4=	-.367879435745E+02	-.367879435745E+02	.3553E-13

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1N(16),Y2N(16),Y3N(16),Y4N(16),EXY1(16),EXY2(16
     1),EXY3(16),EXY4(16),D(3),Q1(6),C(6),V(5),W(5),YP(3)
      OPEN(UNIT = 3, FILE='SLTN2', STATUS = 'NEW')
      D(1)=-0.1
      D(2)=-10.0
      P=D(1)-D(2)
      V(1)=(-0.1-D(2))/P
      V(2)=(-10.0-D(2))/P
      V(3)=(-100.0-D(2))/P
      V(4)=(-1000.0-D(2))/P
      W(1)=(0.1+D(1))/P
      W(2)=(10.0+D(1))/P
      W(3)=(100.0+D(1))/P
      W(4)=(1000.0+D(1))/P
      WRITE(3,99)
      WRITE(3,90)V(1),V(2),V(3),V(4)
      WRITE(3,91)W(1),W(2),W(3),W(4)
      HH=0.5
      DO 31 K=1,2
      Q1(K)=D(K)*HH
      Q=Q1(K)
C     TO EVALUATE Y(X) AND Z(X) USING MULTIDERIVATIVE LINEAR MULT-
C     ISTEP METHOD FOR THE CASE K=2 ORDER 4 ON PROBLEM 5
      APN=1.0+2.0*Q+4.0/3.*(Q**2)+EXP(2.*Q)*(2.0*(Q**2)-3.)/3.0
      APD=Q**2*EXP(2.*Q)-Q*EXP(2.0*Q)+Q*(Q+1.0)
      A=APN/APD
      RPN=1.0+Q+(Q**2)/3.0-EXP(2.*Q)*(1.0-Q+(Q**2)/3.)
      RPD=3.*(Q**2)*(3.*EXP(2.*Q)+1.)/4.-Q*(EXP(3.*Q)-1.)
      R=RPN/RPD
      RQN=1.+(2.-A)*Q+(4./3.-A)*(Q**2)
      RQD=1.-A*Q+(2./3.-A)*(Q**2)
      RQ=RQN/RQD
      WRITE(3,11)Q
      WRITE(3,10)A,R,RQ
      IF(RQ.LT.0.0)GO TO 55
   55 RQ=ABS(RQ)
      T=RQ**1.5
      YN1=R*Q*T+1.+(1.-R)*Q+(1./3.-3*R/4.)*(Q**2)
      YD=1.-Q+(1./3.+9.0*R/4.)*(Q**2)
      YP(K)=YN1/YD
   31 CONTINUE
      WRITE(3,20)YP(1),YP(2)
      Y=1.0
      Z=1.0
      H=0.0
      X=0.0
      DO 30 I=1,10
      H=H+1.0
      Y=Y*YP(1)
      Z=Z*YP(2)
      Y1N(I)=V(1)*Y+W(1)*Z
      Y2N(I)=V(2)*Y+W(2)*Z
      Y3N(I)=V(3)*Y+W(3)*Z
      Y4N(I)=V(4)*Y+W(4)*Z
      WRITE(3,40)H,Y
      X=X+1.0
      EXY1(I)=V(1)*EXP(D(1)*X)+W(1)*EXP(D(2)*X)
      ER1=EXY1(I)-Y1N(I)
      EXY2(I)=V(2)*EXP(D(1)*X)+W(2)*EXP(D(2)*X)
      ER2=EXY2(I)-Y2N(I)
      EXY3(I)=V(3)*EXP(D(1)*X)+W(3)*EXP(D(2)*X)
      ER3=EXY3(I)-Y3N(I)
```

```
      EXY4(I)=V(4)*EXP(D(1)*X)+W(4)*EXP(D(2)*X)
      ER4=EXY4(I)-Y4N(I)
      WRITE(3,65)Y1N(I),EXY1(I),ER1
      WRITE(3,70)Y2N(I),EXY2(I),ER2
      WRITE(3,80)Y3N(I),EXY3(I),ER3
      WRITE(3,75)Y4N(I),EXY4(I),ER4
   30 CONTINUE
   99 FORMAT(5X,'RESULT FOR ORDER 4 h=0.5 ON PROBLEM 5',/)
   90 FORMAT(5X,'V(I)=',4(E12.4,2X)/)
   91 FORMAT(5X,'W(I)=',4(E12.4,2X)/)
   20 FORMAT(5X,'YP(I)=',2F18.12,2X/)
   10 FORMAT(5X,'A R RQ',3(E15.6,2X)/)
   11 FORMAT(5X,' Q = ',2(F10.4,2X)/)
   40 FORMAT(5X,'H= ',F6.3,5X,'Y= ',E18.12,/)
   65 FORMAT(5X,'Y1= ',E18.12,5X,E18.12,5X,E10.4,/)
   70 FORMAT(5X,'Y2= ',E18.12,5X,E18.12,5X,E10.4,/)
   80 FORMAT(5X,'Y3= ',E18.12,5X,E18.12,5X,E10.4,/)
   75 FORMAT(5X,'Y4= ',E18.12,5X,E18.12,5X,E10.4,/)
  100 STOP
      END
```

RESULT FOR ORDER 4 h=0.5 ON PROBLEM 5

V(I)=	.1000E+01	.0000E+00	-.9091E+01	-.1000E+03
W(I)=	.0000E+00	.1000E+01	.1009E+02	.1010E+03

Q = -.0500

A R RQ .100333E+01 .120898E+00 .904837E+00

Q = 5.0000

A R RQ .121662E+01 .315046E+00 .453999E-04

YP(I)= .904837416688 .000045399930

H= 1.000 Y= .904837416688E+00

Y1=	.904837416688E+00	.904837416688E+00	.0000E+00
Y2=	.453999297625E-04	.453999297625E-04	.1605E-16
Y3=	-.822533657183E+01	-.822533657183E+01	.0000E+00
Y4=	-.904791562895E+02	-.904791562895E+02	.0000E+00

H= 2.000 Y= .818730750638E+00

Y1=	.818730750638E+00	.818730750638E+00	.0000E+00
Y2=	.206115362244E-08	.206115362244E-08	.1457E-20
Y3=	-.744300680430E+01	-.744300680430E+01	.0000E+00
Y4=	-.818730748679E+02	-.818730748679E+02	.0000E+00

H= 3.000 Y= .740818217370E+00

Y1=	.740818217370E+00	.740818217370E+00	.0000E+00
Y2=	.935762296883E-13	.935762296884E-13	.9926E-25
Y3=	-.673471106801E+01	-.673471106801E+01	.0000E+00
Y4=	-.740818217481E+02	-.740818217481E+02	.0000E+00

H= 4.000 Y= .670320042040E+00

Y1=	.670320042040E+00	.670320042040E+00	.0000E+00
Y2=	.424835425529E-17	.424835425529E-17	.6008E-29
Y3=	-.609381856492E+01	-.609381856492E+01	.0000E+00
Y4=	-.670320042141E+02	-.670320042141E+02	.0000E+00

```
H=  5.000      Y=  .606530655194E+00
 Y1=   .606530655194E+00      .606530655194E+00      .1110E-15
 Y2=   .192874984796E-21      .192874984796E-21      .3410E-33
 Y3=  -.551391504804E+01     -.551391504804E+01     -.1776E-14
 Y4=  -.606530655285E+02     -.606530655285E+02     -.1421E-13
H=  6.000      Y=  .548811631187E+00
 Y1=   .548811631187E+00      .548811631187E+00      .1110E-15
 Y2=   .875651076268E-26      .875651076270E-26      .1858E-37
 Y3=  -.498919664791E+01     -.498919664791E+01     -.8882E-15
 Y4=  -.548811631270E+02     -.548811631270E+02     -.1421E-13
H=  7.000      Y=  .496585298612E+00
 Y1=   .496585298612E+00      .496585298612E+00      .1665E-15
 Y2=   .397544973590E-30      .397544973591E-30      .9839E-42
 Y3=  -.451441180624E+01     -.451441180624E+01     -.1776E-14
 Y4=  -.496585298686E+02     -.496585298686E+02     -.1421E-13
H=  8.000      Y=  .449328958761E+00
 Y1=   .449328958761E+00      .449328958761E+00      .1665E-15
 Y2=   .180485138784E-34      .180485138785E-34      .5105E-46
 Y3=  -.408480871662E+01     -.408480871662E+01     -.1776E-14
 Y4=  -.449328958828E+02     -.449328958828E+02     -.1421E-13
H=  9.000      Y=  .406569654288E+00
 Y1=   .406569654288E+00      .406569654288E+00      .1665E-15
 Y2=   .819401262396E-39      .819401262399E-39      .2607E-50
 Y3=  -.369608776681E+01     -.369608776681E+01     -.1776E-14
 Y4=  -.406569654349E+02     -.406569654349E+02     -.1421E-13
```

H= 10.000 Y= .367879435690E+00

Y1=	.367879435690E+00	.367879435690E+00	.1665E-15
Y2=	.372007597601E-43	.372007597602E-43	.1315E-54
Y3=	-.334435850677E+01	-.334435850677E+01	-.1332E-14
Y4=	-.367879435745E+02	-.367879435745E+02	-.1421E-13

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