Strong Convergence and Stability of Jungck-Multistep-SP Iteration for Generalized Contractive-Like Inequality Operators

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Abstract
We introduce the Jungck-multistep-SP iteration and prove some convergence as well as stability results for a pair of weakly compatible generalized contractive-like inequality operators defined on a Banach space. As corollaries, the results show that the Jungck-SP and Jungck-Mann iterations can also be used to approximate the common fixed points of such operators. The results are improvements, generalizations and extensions of the work of Chugh and Kumar (2011). Consequently, several results in literature are generalized.

Key words: Jungck-multistep-SP iteration; Zamfirescu (1972).

INTRODUCTION
Most physical systems whose equations are of the form \( f(x) = y \), can be formulated by transforming the equation into a fixed point equation \( x = Tx \) and then apply an approximate fixed point theorem to get information on the existence and uniqueness of fixedpoint, that is, the solution of the original equation. The Picard, Mann, Ishikawa, Noor and multistep iterations have been commonly used to approximate the fixed points of several classes of single quasi-contractive operators. For example see Berinde (2004), Chatterjea (1974), Kannan (1969) and Zamfirescu (1972).

Let \( X \) be a Banach space, \( K \), a nonempty convex subset of \( X \) and \( T : K \rightarrow K \) a self map of \( K \).

**Definition 1.1.** Let \( x_0 \in K \). The Picard iteration scheme \( \{x_n\}_{n=0}^{\infty} \) is defined by
\[
x_{n+1} = Tx_n, \quad n \geq 0
\]
(1.1)

**Definition 1.2.** For any given \( x_0 \in K \), the Mann iteration scheme (Mann, 1953) \( \{x_n\}_{n=0}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n
\]
(1.2)
where \( \{\alpha_n\}_{n=0}^{\infty} \) is a real sequence in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

**Definition 1.3.** Let \( x_0 \in K \). The Ishikawa iteration scheme (Ishikawa, 1974) \( \{x_n\}_{n=0}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n
\]
\[
y_n = (1 - \beta_n)x_n + \beta_n Ty_n
\]
\[
z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n
\]
where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

Observe that if \( \beta_n = 0 \) for each \( n \), then the Ishikawa iteration process (1.3) reduces to the Mann iteration scheme (1.2).

**Definition 1.4.** Let \( x_0 \in K \). The Noor iteration (or three-step) scheme (Noor, 2000) \( \{x_n\}_{n=0}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n
\]
\[
y_n = (1 - \beta_n)x_n + \beta_n Tz_n
\]
\[
z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n
\]
where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

Observe that if \( \gamma_n = 0 \) for each \( n \), then the Noor iteration process (1.4) reduces to the Ishikawa iteration scheme (1.3).

**Definition 1.5.** Let \( x_0 \in K \). The multistep iteration scheme (Rhoades & Soltuz, 2004) \( \{x_n\}_{n=0}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n
\]
\[
y_{n+1} = (1 - \beta_n^p)x_n + \beta_n^p Tz_n, \quad i = 1, 2, ..., p-2
\]
\[
y_{n+1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1} Tx_n, \quad p \geq 2
\]

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where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, i = 1, 2, \ldots, p-1 \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

Observe that the multistep iteration is a generalization of the Noor, Ishikawa and the Mann iterations. In fact, if \( p = 1 \) in (1.5), we have the Mann iteration (1.2); if \( p = 2 \) in (1.5), we have the Ishikawa iteration (1.3) and if \( p = 3 \), we have the Noor iterations (1.4).

One of the most general quasi contractive operators which has been studied by several authors is the Zamfirescu operators.

Suppose \( X \) is a Banach space. The map \( T : X \rightarrow X \) is called a Zamfirescu operator if
\[
||Tx - Ty|| \leq h \max \{||x - y||, ||x - Tx|| + ||y - Ty|| - \frac{||x - Ty|| + d(y - Ty)||}{2}\}
\]
where \( 0 \leq h < 1 \) (Zamfirescu, 1972).

It is known that the operators satisfying (1.6) are generalizations of Kannan maps (Kannan, 1969) and Chatterjea maps (Chatterjea, 1972). Zamfirescu (1972) proved that the Zamfirescu operator has a unique fixed point which can be approximated by Picard iteration (1.1). Berinde (2004) showed that Ishikawa iteration can be used to approximate the fixed point of a Zamfirescu operator when \( X \) is a Banach space while it was shown by Olaleru and Akewe (2006) that if \( X \) is generalised to a complete metrizable locally convex space (which includes Banach spaces), the Mann iteration can be used to approximate the fixed point of a Zamfirescu operator. Several researchers have studied the convergence rate of these iterations with respect to \( X \) is a Banach space.

**Definition 2.1 (Jungck, 1976).** For any \( x_0 \in Y \), there exists a sequence \( \{x_n\}_{n=0}^{\infty} \) such that \( Sx_n = Tx_n \). The Jungck iteration is defined as the sequence \( \{Sx_n\}_{n=1}^{\infty} \) such that
\[
x_{n+1} = Sx_n, \quad n \geq 0
\]
This procedure becomes Picard iteration when \( X = Y \) and \( S = I_Y \) where \( I_Y \) is the identity map on \( X \).

Similarly, the Jungck contraction maps are the maps \( S, T \) satisfying
\[
d(Tx, Ty) \leq k d(Sx, Sy), \quad 0 \leq k < 1 \text{ for all } x, y \in Y
\]
If \( Y = X \) and \( S = I_X \), then maps satisfying (2.2) become the well known contraction maps.

**Definition 2.2 (Singh et al., 2005).** For any given \( u_0 \in Y \), the Jungck-Mann iteration scheme \( \{Sx_n\}_{n=1}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n
\]
where \( \{\alpha_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

**Definition 2.3 (Olatinwo & Imoru, 2008).** Let \( x_0 \in Y \). The Jungck-Ishikawa iteration scheme \( \{Sx_n\}_{n=1}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n
\]
\[
S_y = (1 - \beta_n)x_n + \beta_nTy_n
\]
where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

**Definition 2.4 (Olatinwo, 2008).** Let \( x_0 \in Y \). The Jungck-Noor iteration (or three-step) scheme \( \{Sx_n\}_{n=1}^{\infty} \) is defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n
\]
\[ S_{y_n} = (1 - \beta_n)S_{x_n} + \beta_n T_{x_n} \]  
\[ S_{x_n} = (1 - \gamma_n)S_{x_n} + \gamma_n T_{x_n} \]  
(2.5)

where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \) and \( \{\gamma_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

**Definition 2.5** (Olaleru & Akewe, 2010). Let \( x_0 \in Y \). The Jungck-multistep iteration scheme \( \{S_{x_n}\}_{n=1}^{\infty} \) is defined by

\[ S_{x_{n+1}} = (1 - \alpha_n)S_{x_n} + \alpha_n T_{y_n} \]

\[ S_{y_n} = (1 - \beta_n)S_{y_n} + \beta_n T_{x_n} \]  
(2.6)

where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \) and \( \{\gamma_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

Observe that that the Jungck-multistep iteration is a generalization of the Jungck-Noor, Jungck-Ishikawa and the Jungck-Mann iterations. In fact, if \( k = 2 \) and \( \beta_1 = 0 \) in (2.6), we have the Jungck-Mann iteration (2.3); if \( k = 2 \) in (2.6), we have the Jungck-Ishikawa iteration (2.4) and if \( k = 3 \), we have the Jungck-Noor iterations (2.5).

Observe that if \( X = Y = S = I_p \), then the Jungck-multistep (2.6), Jungck-Noor (2.5), Jungck-Ishikawa (2.4) and the Jungck-Mann (2.3) iterations respectively become the multistep (1.5), Noor (1.4), Ishikawa (1.3) and the Mann (1.2) iterative procedures.

**Definition 2.6** (Chugh & Kumar, 2011). Let \( x_0 \in Y \). The Jungck-SP iteration scheme \( \{S_{x_n}\}_{n=1}^{\infty} \) is defined by

\[ S_{x_{n+1}} = (1 - \alpha_n)S_{x_n} + \alpha_n T_{y_n} \]

\[ S_{y_n} = (1 - \beta_n)S_{y_n} + \beta_n T_{x_n} \]  
(2.7)

where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \) and \( \{\gamma_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

We now consider the following conditions. \( X \) is a Banach space and \( Y \) is a nonempty set such that \( T(Y) \subseteq S(X) \) and \( S: T: Y \rightarrow X \). For \( x, y \in Y \) and \( h \in (0,1) \):

\[ \| Tx - Ty \| \leq h \max \{\| Sx - Sy \|, \frac{\| (Sx-Tx)+(Sy-Ty) \|}{2} \} \]  
(2.8)

\[ \| Tx - Ty \| \leq h \max \{\| Sx - Sy \|, \frac{\| (Sx-Tx)+(Sy-Ty) \|}{2} \}, \| Sx - Ty \| \leq \| Sx - Ty \| \]  
(2.9)

\[ \| Tx - Ty \| \leq \delta \| Sx - Sy \| + L \| Sx - Tx \|, \]  
(2.10)

\[ \| Tx - Ty \| \leq \frac{\delta \| Sx - Sy \| + \delta \| Sx - Ty \|}{1 - M} \]  
(2.11)

\[ \| Tx - Ty \| \leq \delta \| Sx - Sy \| \]  
(2.12)

where \( \phi: R_+ \rightarrow R_+ \) is a monotone increasing sequence with \( \phi(0) = 0 \).

**Remark 2.7.** Observe that if \( X = Y = S = I_p \) (2.8) is the same as the Zamfirescu operator (1.6) already studied by several authors; (2.9) becomes the operator studied by Rhoades (1976); while (2.10) becomes the operator introduced by (Osilike, 1995). Operators satisfying (2.11) and (2.12) were introduced by Olatinwo (2008).

A comparison of the four maps show the following.

**Proposition 2.8** (Olaleru and Akewe, 2010). (2.8) \( \Rightarrow (2.9) \Rightarrow (2.10) \Rightarrow (2.11) \Rightarrow (2.12) \) but the converses are not true. For details of Proof see Olaleru and Akewe (2010).

Bosede (2010) proved some convergence results for the Jungck-Ishikawa and Jungck-Mann iteration processes by using the following more general contractive condition than the Zamfirescu operator

\[ \| Tx - Ty \| \leq e^{L/2} \| Sx - Sy \| + 2L \| Sx - Tx \|, \]  
(2.13)

for all \( x, y \in Y \) where \( L \geq 0 \). Motivated by the work of Bosede (2010), Chugh and Kumar (2011), introduced the following contractive-like inequality operators and proved strong convergence and stability results for the Jungck-SP iterative scheme (2.7).

\[ \| Tx - Ty \| \leq e^{L/2} \| Sx - Sy \| + 2L \| Sx - Tx \|, \]  
(2.14)

for all \( x, y \in Y \) where \( L \geq 0 \) and \( \phi: R_+ \rightarrow R_+ \) is a monotone increasing sequence with \( \phi(0) = 0 \).

However, inspired by the work of Chugh & Kumar (2011), we introduce the following Jungck-multistep-SP and use it to approximate the common fixed point using the contractive condition (2.14).

**Definition 2.9.** Let \( x_0 \in Y \). The Jungck-multistep-SP iterative process \( \{S_{x_n}\}_{n=0}^{\infty} \) is defined by

\[ S_{x_{n+1}} = (1 - \alpha_n)S_{x_n} + \alpha_n T_{y_n} \]

\[ S_{y_n} = (1 - \beta_n)S_{y_n} + \beta_n T_{x_n} \]  
(2.15)

where \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \) are real sequences in \([0,1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

Observe that that the Jungck-multistep iteration is a generalization of the Jungck-Ishikawa iteration (2.5). We need the following definition.

**Definition 2.10** (Abbas and Jungck, 2008). A point \( x \in X \) is called a coincidence point of a pair of self maps \( S, T \) if there exists a point \( w \) (called a point of coincidence) in \( X \) such that \( w = Sx = Tx \). Self-maps \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence points, that is, if \( Sx = Tx \) for some \( x \in X \), then \( STx = TStx \).

Chugh and Kumar (2011) proved that the Jungck-SP scheme converges to the coincidence point of \( S, T \) defined by (2.14) when \( S \) is an injective operator. It was shown in Olatinwo (2008) that the Jungck-Ishikawa iteration converges to the coincidence point of \( S, T \) defined by (2.12) when \( S \) is an injective operator while the same convergence result was proved for Jungck-Noor when \( S, T \) are defined by (2.11) (Olatinwo, 2008). (We note that the maps satisfying (2.9) and of course (2.10)-(2.14) need not have a coincidence point (Olaleru & Akewe, 2010)). We rather prove the convergence of Jungck-multistep-SP iteration (2.15) to the unique common fixed point of \( S, T \) defined by
(2.14), without assuming that $S$ is injective, provided the coincidence point exist for $S,T$.

**Lemma 2.11 (Berinde, 2004):** Let $\delta$ be a real number satisfying $0 \leq \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$ then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$, satisfying $u_{n+1} \leq \delta u_n + \varepsilon_n$, $n = 0,1,2,\ldots$, we have $\lim_{n \to \infty} u_n = 0$.

**MAIN RESULTS**

**Theorem 3.1.** Let $X$ be a Banach space and $S,T: Y \to X$ for an arbitrary set $Y$ such that (2.14) holds and $T(Y) \subseteq S(Y)$.

Assume $S$ and $T$ have a coincidence point $z$ such that $Tz = Sz = p$. For any $x_0 \in Y$, the Jungck-multistep-SP iterative process (2.15) $\{S_{x_0}^n\}_{n=0}^{\infty}$ converges strongly to $p$.

Further, if $Y = X$ and $S,T$ commute at $p$ (i.e. $S$ and $T$ are weakly compatible), then $p$ is the unique common fixed point of $S,T$.

**Proof.** In view of (2.14) and (2.15) coupled with the fact that $Tz = Sz = p$, we have

$$\|S_{x_0}^{n+1} - p\| \leq (1 - \alpha_n) \|S_{x_0}^n - p\| + \alpha_n \|T_{x_0}^n - p\|$$

$$\leq (1 - \alpha_n) \|S_{x_0}^n - p\| + \alpha_n \|Tz - T_{x_0}^n\|$$

$$\leq (1 - \alpha_n) \|S_{x_0}^n - p\| + \alpha_n \|Tz - Tz_{x_0}^n\| + \phi(\delta \|Sz - Tz\|)$$

$$= (1 - \alpha_n) \|S_{x_0}^n - p\| + \delta \|Sz - Tz\|$$

$$= [1 - \alpha_n(1 - \delta)] \|S_{x_0}^n - p\|$$

An application of (2.15) and (2.14) also give

$$\|S_{x_0}^{n+1} - p\| \leq (1 - \beta_n) \|S_{x_0}^n - p\| + \beta_n \|T_{x_0}^n - p\| + \phi(\delta \|Sz - Tz\|)$$

$$= (1 - \beta_n) \|S_{x_0}^n - p\| + \beta_n \|Tz - T_{x_0}^n\| + \phi(\delta \|Sz - Tz\|)$$

$$= [1 - \beta_n(1 - \delta)] \|S_{x_0}^n - p\|$$

Similarly, an application of (2.15) and (2.14) also give

$$\|S_{x_0}^{n+1} - p\| \leq (1 - \beta_n) \|S_{x_0}^n - p\| + \beta_n \|T_{x_0}^n - p\| + \phi(\delta \|Sz - Tz\|)$$

$$= [1 - \beta_n(1 - \delta)] \|S_{x_0}^n - p\|$$

Continuing the above process we have

$$\|S_{x_0}^{n+1} - p\| \leq [1 - \alpha_n(1 - \delta)] \|S_{x_0}^n - p\| \leq [1 - \beta_n(1 - \delta)] \|S_{x_0}^n - p\| \leq \cdots \leq [1 - \beta_n^{n+1}(1 - \delta)] \|S_{x_0}^n - p\|$$

An application of (2.15) and (2.14) also give

$$\|S_{x_0}^{n+1} - p\| \leq [1 - \alpha_n(1 - \delta)] \|S_{x_0}^n - p\| \leq [1 - \beta_n(1 - \delta)] \|S_{x_0}^n - p\| \leq \cdots \leq [1 - \beta_n^{n+1}(1 - \delta)] \|S_{x_0}^n - p\|$$

Thus

$$\|S_{x_0}^{n+1} - p\| \leq e^{-\delta \sum_{j=0}^{n} \alpha_j} \|S_{x_0}^n - p\|$$

Since $0 \leq \delta < 1$, $\alpha_j \in (0,1)$ and $\sum_{j=0}^{n} \alpha_j = \alpha_n$, so

$$\|S_{x_0}^{n+1} - p\| \to 0$$

as $n \to \infty$.

Thus, $\lim_{n \to \infty} \|S_{x_0}^{n+1} - p\| = 0$.

Therefore, $\{S_{x_0}^n\}_{n=0}^{\infty}$ converges strongly to $p$.

Next we show that $p$ is unique. Suppose there exists another point of coincidence $z^*$. Then there is an $z^* \in X$ such that $Tz^* = Sz^* = p$. Hence, using (2.14) we have

$$\|z - z^*\| = \|Tz - Tz^*\|$$

$$\leq e^{\delta \|Sz - Sx_0\|} + \phi(\|Sz - Tz\|)$$

$$= \delta \|z - z^*\|$$

Since $\delta < 1$, then $z = z^*$ and so $p$ is unique.

Since $S,T$ are weakly compatible, then $TSz = STz$ and so $Tz = Sp$. Hence $p$ is a coincidence point of $S,T$ and since the coincidence point is unique, then $p = z$ and hence $Sp = Tp = p$ and therefore $p$ is the unique common fixed point of $S,T$. This ends the proof.

Theorem 3.1 leads to the following Corollaries:

**Corollary 3.2.** Let $X$ be a Banach space and $S,T: Y \to X$ for an arbitrary set $Y$ such that (2.14) holds and $T(Y) \subseteq S(Y)$. Assume $S$ and $T$ have a coincidence point $z$ such that $Tz = Sz = p$. For any $x_0 \in Y$, the Jungck-SP iterative process (2.7) $\{S_{x_0}^n\}_{n=0}^{\infty}$ converges strongly to $p$.

Further, if $Y = X$ and $S,T$ commute at $p$ (i.e. $S$ and $T$ are weakly compatible), then $p$ is the unique common fixed point of $S,T$.

**Corollary 3.3.** Let $X$ be a Banach space and $S,T: Y \to X$ for an arbitrary set $Y$ such that (2.14) holds and $T(Y) \subseteq S(Y)$. Assume $S$ and $T$ have a coincidence point $z$ such that $Tz = Sz = p$. For any $x_0 \in Y$, the Jungck-Mann iterative process (2.3) $\{S_{x_0}^n\}_{n=0}^{\infty}$ converges strongly to $p$. 

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Further, if \( Y = X \) and \( S, T \) commute at \( p \) (i.e. \( S \) and \( T \) are weakly compatible), then \( p \) is the unique common fixed point of \( S, T \).

**Remark 3.4.** Weaker versions of Theorem 3.1 are the convergence results in Chugh and Kumar (2011) where \( S \) is assumed injective and the convergence is not to the common fixed point but to the coincidence point of \( S, T \).

Furthermore, the Jungck-multistep-SP iteration used in Theorem 3.1 is more general than the Jungck-SP used in Chugh and Kumar (2011).

### STABILITY OF JUNGCK-MULTISTEP-SP ITERATIONS IN A BANACH SPACE

In this section, some stability results for the Jungck-multistep-SP iterative processes defined by (2.15) are established for generalized contractive-like inequality operators defined by (2.14). The stabilities of Jungck-SP and Jungck-Mann iterative processes follow as corollaries.

The theorem is stated thus:

**Theorem 4.1.** Let \( X \) be a Banach space and \( S, T : Y \rightarrow X \) for an arbitrary set \( Y \) such that (2.14) holds and \( T(Y) \subseteq S(Y) \). For any \( x_n \in Y \) and \( 0 \leq \delta < 1 \), let \( \{ Sx_n \}_{n=0}^{\infty} \) be the Jungck-multistep-SP iterative process defined by (2.15) converging to \( p \) (that is \( S^p = T^p = p \)) with \( 0 < \alpha < \alpha_n, 0 < \beta < \beta_n \) for \( i = 1, 2, \ldots, k-1 \) and all \( n \). Then the Jungck-multistep-SP iterative process defined by (2.15) is \((S, T)\)-stable.

**Proof.** Let \( \{ Sx_n \}_{n=0}^{\infty} \subseteq Y \), \( \{ Sx_n \}_{n=0}^{\infty} \subseteq Y \), for \( i = 1, 2, \ldots, k-1 \) be real sequences in \( Y \).

Let \( \varepsilon_n = \| Sx_{n+1} - (1 - \alpha_n)Sx_n - \alpha_n Tz_n \|, n = 0, 1, 2, \ldots \), where

\[
Sx_n = (1 - \beta_n)Sx_{n+1} + \beta_n Tz_n
\]

for \( i = 1, 2, \ldots, k-2 \),

\[
Sx_n = (1 - \beta_n)^{i-1}Sx_n + \beta_n^{i-1}Tz_n, \quad i = 1, 2, \ldots, k-2
\]

and let \( \lim_{n \rightarrow \infty} \varepsilon_n = 0 \).

Then, we shall prove that \( \lim_{n \rightarrow \infty} Sx_n = p \) using the generalized contractive-like inequality operators satisfying condition (2.14).

That is,

\[
\| Sx_{n+1} - p \| \leq (1 - \alpha_n)Sx_n - \alpha_n Tz_n \| + \| Sx_n - p \| + \alpha_n Tz_n - Tz_n \| \leq \varepsilon_n + (1 - \alpha_n)\| Sx_n - p \| + \alpha_n \| Tz_n - Tz_n \| \leq \varepsilon_n + (1 - \alpha_n)\| Sx_n - p \| + \alpha_n \| p - Sx_n \| \leq \varepsilon_n + (1 - \alpha_n)\| Sx_n - p \| + \alpha_n e^{1\| \| Sx_n - p \| + \alpha_n \| p - Sx_n \|} (4.1)
\]

Substituting (4.1) in (4.2), we have

\[
\| Sx_{n+1} - p \| \leq (1 - \beta_n)^{i-1}Sx_n - \beta_n^{i-1}Tz_n \| + \beta_n^{i-1}e^{1\| \| Sx_n - p \| + \alpha_n \| p - Sx_n \|} (4.3)
\]

Similarly, an application of (2.15) and (2.14) give

\[
\| Sx_{n+1} - p \| \leq (1 - \beta_n)^{i-1}Sx_n - \beta_n^{i-1}Tz_n \| + \beta_n^{i-1}e^{1\| \| Sx_n - p \| + \alpha_n \| p - Sx_n \|} (4.4)
\]

Combining (4.1), (4.2), (4.3) (4.4) and (4.5), we have

\[
\| Sx_{n+1} - p \| \leq (1 - \beta_n^\delta)(1 - \beta_n^\delta)\| Sx_{n-1} - p \| + \beta_n e^{1\| \| Sx_n - p \| + \alpha_n \| p - Sx_n \|} (4.6)
\]

Substituting (4.7) in (4.6), we have

\[
\| Sx_{n+1} - p \| \leq (1 - \beta_n^\delta)(1 - \beta_n^\delta)\| Sx_{n-1} - p \| + \beta_n e^{1\| \| Sx_n - p \| + \alpha_n \| p - Sx_n \|} (4.8)
\]

Using 0 \( \leq \alpha_n \leq \beta_n \) and \( \delta < 1 \), we have

\[
[1 - \alpha_n (1 - \delta)][1 - \beta_n^\delta (1 - \delta)][1 - \beta_n^\delta (1 - \delta)] < 1
\]

Using Lemma (2.11), (4.8) yields \( \lim_{n \rightarrow \infty} Sx_n = p \). Conversely, let \( \lim_{n \rightarrow \infty} Sx_n = p \), we show that \( \lim_{n \rightarrow \infty} \varepsilon_n = 0 \) as follows:
\[ e_n = \| S_{\alpha_n} - (1-\alpha_n)T_{\alpha_n} - \alpha_nT_{\alpha_n} \| \]
\[ \leq \| S_{\alpha_n} - p \| + \| (1-\alpha_n) + \alpha_n \| \| (1-\alpha_n)S_{\alpha_n} - \alpha_nT_{\alpha_n} \| \]
\[ \leq \| S_{\alpha_n} - p \| + (1-\alpha_n) \| S_{\alpha_n} - p \| + \alpha_n \| T_{\alpha_n} - T_{\alpha_n} \| \quad (4.9) \]

However,
\[ \| S_{\alpha_n} - p \| \leq [1-\beta_n(1-\delta)] [1-\beta_n(1-\delta)] [1-\beta_n(1-\delta)] \]
\[ \text{Substituting (4.10) in (4.9), we have} \]
\[ e_n \leq [1-\alpha_n(1-\delta)] [1-\beta_n(1-\delta)] [1-\beta_n(1-\delta)] \]
\[ \| S_{\alpha_n} - p \| \]
\[ \text{Since } \lim_{n \to \infty} e_n = 0 \text{ (by our assumption), it follows that} \]
\[ \lim_{n \to \infty} e_n = 0, \]

Therefore the Jungck-multistep-SP iterative scheme (2.15) is \((S,T)\)-stable.

**Corollary 4.2.** Let \( X \) be a Banach space and \( S,T : Y \to X \) for an arbitrary set \( Y \) such that (2.14) holds and \( T(Y) \subseteq S(Y) \). For any \( x_0 \in Y \) and \( 0 < \delta < 1 \), let \( \{x_n\}_{n=0}^\infty \) be the Jungck-SP iterative process defined by (2.7) converging to \( p \) (that is \( S^p = T^p = p \)) with \( 0 < \alpha < \alpha_0 \), and all \( n \). Then the Jungck-SP iterative process defined by (2.7) is \((S,T)\)-stable.

**Corollary 4.3.** Let \( X \) be a Banach space and \( S,T : Y \to X \) for an arbitrary set \( Y \) such that (2.14) holds and \( T(Y) \subseteq S(Y) \). For any \( x_0 \in Y \) and \( 0 < \delta < 1 \), let \( \{S_n\}_{n=0}^\infty \) be the Jungck-Mann iterative process defined by (2.3) converging to \( p \) (that is \( S^p = T^p = p \)) with \( 0 < \alpha < \alpha_0 \), and all \( n \). Then the Jungck-Mann iterative process defined by (2.3) is \((S,T)\)-stable.

**Remark 4.4.** Weaker versions of Theorem 4.1 are the stability results in Chugh and Kumar (2011) where \( S \) is assumed injective and the stability result is not to the common fixed point but to the coincidence point of \( S,T \). Furthermore, the Jungck-multistep-SP iteration used in Theorem 4.1 is more general than the Jungck-SP used in Chugh and Kumar (2011).

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**References**


