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TOPIC:

NUMBERS: THE LANGUAGE OF THE THINKERS

By

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NUMBERS:
THE LANGUAGE OF THE THINKERS

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Members of Senate,
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Dear students,
My family and friends,
Distinguished guests,
Ladies and Gentlemen

Preamble
I give all glory and honour to the Almighty God who made it possible for me to stand before this audience to deliver this inaugural lecture. I stand before you this day because the University found me worthy to be elevated to the post of a professor of Mathematics.

Since my appointment I have thought of what will be the title of my inaugural lecture and while toiling with this, my mentor said, “you have worked very well with numbers why not consider something along that line”. Also I have jointly written more papers with him than anyone else. For these reasons I hereby dedicate this inaugural lecture to my mentor and academic father, Late Professor Adetokunbo Babatunde Sofoluwe, whom God used greatly in my life for greater part of my academic pursuit, and this helped me to where I am today. May his soul rest in peace.

I will not want to emphasise the need for newly appointed professors to give inaugural lecture, as this is a requirement that must be satisfied. To this end I decided to give my inaugural lecture this day as the 6th lecture series in 2012/2013 session.
Prof A. B. Sofoluwe delivered his inaugural lecture on the 8th of November 2006 on a title, “Beyond Calculations”. I observed that calculations are done everyday almost by every grown up persons unawares. These calculations are done not with any other thing other than numbers, however, no one thinks or border about the everyday calculation or usage of numbers that we engage in. This prompt me to title my own lecture as “Numbers: the Language of the Thinkers”.

Introduction
Mathematics has been perceived as a difficult subject in schools. Young and old, adult and school children perceive Mathematics as a subject that is not easy to grasp or retain. The difficulty perceived by many is not far-fetched. It is simply because Mathematics talks with symbols. It is quite clear that Mathematics has its own language just as there are languages being spoken all over the world for the purpose of communication. Everything about Mathematics started from numbers. In fact, there are several types of numbers existing in the mathematical world. No man on the earth can avoid dealing with numbers such as 1,2,3,...,9,0. These numbers are well known as they may appear, they are merely symbols with attached values but the use of symbols seems to be the beginning of problems people have with Mathematics.

It is suffice to say that Mathematics unlike other languages, speaks “only the truth and nothing but the truth”. It does not allow you to change truth to false or vice-versa. Whatever is mathematically true today remains true tomorrow and nothing can be done to it. For example, the number 4 will always be greater than the number 3. Without compounding the problem for those who cannot speak Mathematics language, we simply write $4 > 3$ or $3 < 4$. These two symbols being introduced here can give a nightmare to someone. However, these symbols are right there on the typewriter or computer keyboard which do not frighten while typing them, but may look inconvenient to some people when constants or numbers are included such as $4 > 3$ or $a > 5$ or even $y < 1$. The above examples simply gave us the foundation that Mathematics has its language. The
purpose of this lecture is to prove to my audience that as long as you are born into this world you have everything to do with Mathematics and in particular with numbers, and secondly that Mathematics is a friend to all.

There are different levels of friendship existing among people. Some people are your friends because you work in the same place or you attend the same school or belong to the same class. Some other develop closer friendships and do visit each other at home or become family friends. Some are friends because they have common faith. They are friends because they attend the same church or worship in the same place. Some are also friends because they belong to the same social club. Whatever it may be different degrees of friendship exist. In the same vein, everyone has to use numbers or can I say everyone develops some degree of friendship with numbers, then a higher degree of friendship with everyday Arithmetic or Mathematics as the case may be, and fewer numbers of people have deeper friendship with higher Mathematics while the minority have true friendship with what is called Abstract Mathematics.

Although, Mathematics would necessarily include numbers, we can say that arithmetic is the mother of numbers, while Mathematics is the father of arithmetic. In other words, Mathematics contain Arithmetic and some other things, while Arithmetic itself is more than just numbers but putting those numbers together for use by introducing some mathematical operations such as addition, subtraction and so on.

This lecture will attempt not to leave my audience behind, as this simple lecture cannot make you a mathematician in a day. It will however create in you all the essence of living and how impossible it is for any man to exist without numbers and Mathematics, because numbers happen to be everybody's business.
Types of Numbers
There are various types of numbers. Numbers can broadly be classified into two. There are real numbers and there are complex numbers.

The real numbers are as real as what you can appreciate in terms of concrete value or quantity, while the complex numbers are not so, they contain some imaginary numbers that may not be easily conceptualized. Complex numbers are not complex in terms of complexity as such, but in terms of creating another world of numbers which may not directly be used by all. They however have relevance in applications and in drawing conclusion from a mathematical problem.

For this simple reason we may just concentrate on real numbers. Real numbers are in various stages and of different categories. We have the following types or classification of numbers:

- The counting numbers
- The natural numbers
- The integers
- The rational numbers
- The irrational numbers
- The real numbers, and
- The complex numbers

Apart from the last one, we often use all these numbers directly or indirectly, without knowing them. In fact numbers are better appreciated if one can master the classification, as this will help to put things right. Let me illustrate with this simple mistake which is common in our schools today, and which many are guilty of. We all know decimal numbers. Some of them are rational numbers and some are not. A number like 25.47 is pronounced in different ways because of poor understanding of decimal numbers. A number that is greater than 0 but less than 1 can be expressed as a decimal number, however such numbers that follow the decimal point is not pronounced together as if they are greater than ten. Thus 25.47 is pronounced as twenty-five point four seven, and not
twenty-five point forty-seven, neither is it right to call it as twenty-five point forty-seven. The 4 and the 7 after the decimal point is pronounced one by one, and not as a single number greater than 10.

What about the mobile phone number? Many people always pronounce the numbers in an unfriendly and non-numeric manner. A phone number such as 08023043270 is sometimes pronounced as 08023043270. All the zeros are pronounced as letter “O” instead of number “0” (zero).

Figure 1: Tree diagram of Numbers

It is assumed that almost all the audience present here have used real numbers and their subsets before now. You may not remember them now, but the fact is that any time you need them you do make use of them. The use of these numbers will normally require a man to think before taking decision.
The Historical background
Mathematics has always been part of human being from the day the first man landed on earth. As far as 3500BC, man has been using numbers to count. The shepherd cannot know the number of cattle he has except he uses numbers, or at least use a form of tally of things to keep his records. In the process of counting, he may use his fingers and toes, or make marks on the wall, or use broom sticks and so on. This completely shows that man cannot exist without numbers. Thus we can say that the concept of numbers is the heart of Mathematics from which higher mathematical concepts developed. [Russ Rowlett (2003)].

Interestingly, everybody needs number because it is the language for the living. Different people of different language background may call the number 100 different names, but the value of what they all may be saying in different languages will be the same. All of them will be referring to the same size or quantity. Hence Mathematics is universal.

I have, received in the past, series of questions from school children and students about numbers because they get frightened with Mathematics. Some of the questions are: what are numbers? Who started or introduced Mathematics? Who needs Mathematics? Where does Mathematics and numbers come from? Why do we need negative numbers, when positive numbers are more than enough headaches to handle? Is infinity a number and if so what is its value? And several other questions like these ones. All of these have simple answer.

Numbers can be said to be figures or symbols given to a thing as its value or quantity so as to appreciate its worth or size. Of course, man did not invent numbers or Mathematics. Man only discovered that numbers are in existence and began to give names to them as he discovers more numbers. For example, in the early days, there is no number called a million. Everything is counted in tens, hundreds and later in thousands. But after a while, man did not only need larger numbers but move from millions to billions and then to trillions. Of course the names were given to these numbers but that is not without
some difficulties and universal agreement between users of numbers.

You will observe the following numbers:

\[
\begin{align*}
10 &= 1 \times 10 \\
100 &= 10 \times 10 \\
1,000 &= 100 \times 10 \\
1,000,000 &= 1,000 \times 1,000 \\
1,000,000,000 &= 1,000,000 \times 1,000 \\
1,000,000,000,000 &= 1,000,000,000 \times 1,000
\end{align*}
\]

You will notice that at the earlier and smaller numbers, the name changed after a multiple of ten. However, when we get to a million it is a thousand times a thousand. From there on, the name changes after a multiple of a thousand. Thus a million changed to a billion and a billion to a trillion. From this idea, we already know the next level of numbers as a trillion times a thousand. Numbers starting from a billion is rather confusing and does not have a general agreement, especially as some are termed as American style while another European. In all of this confusion of high numbers as per the name, it is out of place to ascribe the invention of a billion or trillion to one Mr X or Y. The scientific truth is that numbers have been in existence before man's existence, so man could not have invented numbers. The only option left to man is to agree that God is the creator and the inventor of numbers and Mathematics.

**Names for Large Numbers**

The English names for large numbers are coined from the Latin names for small numbers \(n\) by adding the ending \(-illion\), suggested by the name "million." Thus billion and trillion are coined from the Latin prefixes \(bi-\) \((n = 2)\) and \(tri-\) \((n = 3)\), respectively. In the American system for naming large numbers, the name coined from the Latin number \(n\) applies to the number \(10^{3n+3}\). In a system traditional in many European countries, the same name applies to the number \(10^{6n}\). In particular, a billion is \(10^9 = 1,000,000,000\) in the American
system and $10^{12} = 1\,000\,000\,000\,000$ in the European system. For $10^9$, Europeans say "thousand million" or "milliard."

Although we describe the two systems today as American or European, both systems are actually of French origin. The French physician and mathematician Nicolas Chuquet (1445-1488) apparently coined the words *byllion* and *tryllion* and used them to represent $10^{12}$ and $10^{18}$, respectively, thus establishing what we now think of as the "European" system. However, it was also French mathematicians of the 1600's who used *billion* and *trillion* for $10^9$ and $10^{12}$, respectively. This usage became common in France and in America, while the original Chuquet nomenclature remained in use in Britain and Germany. The French decided in 1948 to revert to the Chuquet ("European") system, leaving the U.S. as the chief standard bearer for what then became clearly an American system.

In recent years, American usage has eroded the European system, particularly in Britain and to a lesser extent in other countries. This is primarily due to American finance, because Americans insist that $1,000,000,000$ be called a billion dollars. In 1974, the government of Prime Minister Harold Wilson announced that henceforth "billion" would mean $10^9$ and not $10^{12}$ in official British reports and statistics. The *Times of London* style guide now defines "billion" as "one thousand million, not a million million." There is yet a greater number that its value is unimaginable. This number is called infinity ($\infty$). Infinity is a number that is so large that value cannot be assigned to it.
Everyday Usage of Mathematics and Numbers

Numbers are limited when they are not translated into mathematical tools. We observed that Mathematics has been a tool used on daily basis to achieve the day’s task. For example, if you are to buy a car, follow a recipe, go to work or decorate your home, you are bound to use Mathematics principles. People have been using these same principles for thousands of years, across countries and continents. Whether you are sailing a boat off the West African coast or driving on a bend road or building a house in Abuja or Yenogoa, you are using Mathematics to get things done.

How can Mathematics be so universal? First, since human beings did not invent Mathematics concepts; but only discovered them, then the language of Mathematics is not English, French or Yoruba, neither is it German or Russian, though there are great mathematicians of old who were Russians and Germans. The language of Mathematics is number. If we are well versed in this language of numbers, it can help us to think rightly, make important decisions and perform everyday tasks. Thus, numbers or Mathematics is the language of the thinkers who is set to make a right decision. Mathematics can help one to shop wisely, spend prudently,
buy the right insurance, remodel a home within a budget, understand population growth, or even bet on the football team with the best chance of winning a match.

Numbers are so important that we hardly exist without it. So much has been criticized of the difficulty that Mathematics poses to man, but the truth of the matter is that everyday living requires Mathematics. That reminds me of the story of Chinedu who was helping his uncle in a shop to sell vehicle spare parts. While in the class, the teacher asked Chinedu few questions such as; 20 times 8, Chinedu did not know the answer; 50 plus 120, Chinedu missed this sum, and finally 200 minus 60, he could not provide the correct answer. He was reported to the school's Principal who understood very well the background of this boy. The school's Principal said, “Chinedu, I bought goods of N60 and gave you N200”, what is my change. He quickly answered, “N140 sir. What of N50 plus N120”, he said “N170 sir”, and finally the Principal said, “how much do you sell rear light bulb of a motor car in your shop”, he said “N20”. “If I buy 8 of them, how much will I pay you?” He said, “N160 sir”. Then the teacher understood that Chinedu's kind of number or Mathematics is money-wise and not ordinary numbers.

Let me take you through the lane on how Mathematics can help us in our daily lives. First of all you need to look at the language of numbers through common situations, such as playing games or cooking. Put your decision-making skills to the test by deciding whether buying or leasing a new car is right for you, and predict how much money you can save for your retirement.

What about increasing your savings, you surely need some Mathematics understanding of simple and compound interests. The principles of simple and compound interest are the same whether you are calculating your earnings from a savings account or your gains from shares on stocks. Paying a little attention to these principles could mean big payoffs over time.
Mathematics on Phone calls
In everyday phone calls, you want to flow along with the network provider and monitor what is the best offer. From MTN to Airtel or Globacom, you are made to create ten friends, call it family and friends. They vary the offer so that you can decide what plan you want to join. You have some 7.5k/sec for family and friends and 20k/sec for other networks or 10k/sec for family and friends and 15k/sec for other networks. You need a little but simple Mathematics to decide which one will pay off. This simply suggests that you will also need some thinking to arrive at the best offer. These network providers are forcing you to think as they are playing with numbers. So we can once again say that “Numbers” is the language of the thinkers.

Mathematics of Interest on Savings Account
In banking, interest is calculated and added at the end of a certain time period. You might have a savings account that offers a 3% interest rate annually. At the end of each year, the bank multiplies the principal (the amount in the account) by the interest rate of 3% to compute what you have earned in interest. If the interest will drop in your account at a particular date, you must ensure that you do not withdraw from that account before that day so as to earn your full interest.

Areas of Rectangles and Curves
Mathematics is equally useful in home decorating. Most home decorators need to work within a budget. But in order to figure out what you will spend, you first have to know what you need. How will you know how many rolls of wallpaper or number of yards or metre of decorating cloth to buy if you do not calculate how much wall space you have to cover? Understanding some basic geometry can help you stick to your budget. If your walls are mainly rectangles then it is easier to measure and calculate the amount of cloth needed for the decoration. However, if the walls include a curve then you will need further knowledge of area of a circle which will involve the use of pi. Pi is a constant relating to diameter of a circle and assumes an approximate value of 22/7 or 3.14159
Imagine you are planning to buy new carpeting for your home (see Figure 2a). You are going to put down carpeting in the living room, bedroom, and hallway, but not in the bathroom. A simple Mathematics shows that the whole house will need a dimension of 264 sq metres (that is, $22 \times 12$). However removing the bathroom from this figure we obtain 229 sq.m.

If your living room has a semi-circular alcove as shown in the floor plan of Figure 2 above, you will need to use additional formula of area of a circle to find its area. A calculation of this alcove part gives 56.5 square metres and this can be added to the earlier floor plan's area of 229 square metres to get the total area you want to carpet as 285.5 square metres. Thus by using geometry, you can buy exactly the amount of carpet you need.

**Numbers needed for cooking**

In following a recipe for food and edibles you will agree with me that not all people are chefs, but we are all eaters. Most of us need to learn how to follow a recipe at some point. To create dishes with good flavour, consistency, and texture, the various ingredients must have a kind of relationship to one another. For instance, to make cookies that both look and taste
like cookies, you need to make sure you use the right amount of each ingredient. Add too much flour and your cookies will be solid as a rock. Add too much salt and they will taste terrible. The way out of the trouble of not having too much of a particular ingredient is to have a knowledge of mathematical term called ratios. To know that ingredients have relationships to each other in a recipe is an important concept in cooking. It is also an important Mathematics concept of ratio. If a recipe calls for 1 egg and 2 cups of flour, the relationship of eggs to cups of flour is 1 to 2. In mathematical language, that relationship can be written in two ways: \( \frac{1}{2} \) or 1:2. Both of these express the ratio of eggs to cups of flour: 1 to 2. If you mistakenly alter that ratio, the results may not be edible.

Mathematics and Numerical Analysis

Mr. Vice-Chancellor sir, as one advances in the study of Mathematics, various symbols are being introduced in other to solve many real life problems. For example a problem is posed on the output of a company as follows:
The square of the present output of a company equals four times the output in the previous year plus a constant sum of five. What is the output of the company?

This scenario can be translated into a mathematical equation by writing
\[ x^2 = 4x + 5 \]

This is then solved by writing
\[ x^2 - 4x - 5 = 0 \quad \text{or} \quad (x - 5)(x + 1) = 0 \]
Which gives \( x = 5 \) or -1

Now we may have some equations of this kind which may not be easily solved the way this is done as many real life problems may not be as exact in figures as the coefficients of \( x \) looks like in this case. In real life we may have some equations describing a phenomenon which may look like
This equation will require the use of a quadratics formula and sometimes may be cumbersome. An alternative approach is to introduce an approximate method which can get the roots of this equation with a value very close to the exact solution. The process of using the alternative and approximate method is another branch of Mathematics called the Numerical Analysis, which is my main area of research. What then is Numerical Analysis?

**Numerical Analysis**

Numerical analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from Discrete Mathematics).

Numerical analysis is also defined as the area of Mathematics and Computer science that creates, analyzes, and implements algorithms for solving numerically the problems of continuous Mathematics.

One of the earliest mathematical writings is a Babylonian tablet from the Yale Babylonian Collection (YBC 7289), which gives a sexagesimal numerical approximation of $\sqrt{2}$, the length of the diagonal in a unit square. Being able to compute the sides of a triangle (and hence, being able to compute square roots) is extremely important, for instance, in carpentry and construction. Thus an approximate method is sort to find the value of $\sqrt{2}$

The square root of 2 is approximated by sum of fractions as

$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{602} + \frac{10}{603} = 1.41421296...$

Numerical analysis continues this long tradition of practical mathematical calculations. Much like the Babylonian approximation of $\sqrt{2}$, numerical analysis does not seek exact answers, because exact answers are often impossible to obtain in practice. Instead, much of numerical analysis is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors.
A numerical method is the same as an algorithm, the steps required to solve a numerical problem. Algorithms became very important as computers were increasingly used to solve problems. It was no longer necessary to solve complex mathematical problems with a single closed form equation. Numerical analysis naturally finds applications in all fields of engineering and the physical sciences and in the 21st century, the life sciences and even the arts have adopted elements of scientific computations. Ordinary differential equations appear in the movement of heavenly bodies (planets, stars and galaxies); numerical linear algebra is important for data analysis.

Before the advent of modern computers, numerical methods often depended on hand interpolation in large printed tables. Since the mid 20th century, computers calculate the required functions instead. These same interpolation formulas nevertheless continue to be used as part of the software algorithms for solving differential equations. (Wikipedia, the free encyclopaedia, 2004)

Numerical analysis is a mathematical technique that is used to handle problems which originate generally from real-world applications of algebra, geometry, and calculus, and they involve variables which vary continuously. These problems occur throughout the natural sciences, social sciences, medicine, engineering, and business. Beginning in the 1940's, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science, medicine, engineering, and business; and numerical analysis of increasing sophistication has been needed to solve these more accurate and complex mathematical models of the world. The formal academic area of numerical analysis varies from highly theoretical mathematical studies to computer science issues involving the effects of computer hardware and software on the implementation of specific algorithms (Atkinson, 2007).
Today, numerical analysis now plays a central role in engineering and in the quantitative parts of pure and applied science. The tasks of numerical analysis include the development of fast and reliable numerical methods together with the provision of a suitable error analysis. The algorithms are developed as computer programs, taking full account of machine architectures such as parallelism (Daintith, 2004). Thus, numerical analysis involves development of methods and determines the performance of a numerical method being used to solve a problem. It also examines its accuracy, stability and convergence and in general, the efficiency of the algorithm with respect to its order.

Areas of Numerical Analysis
Numerical analysis has grown so large and wide that it can be categorized into various classes because of the kind of problems they handle and the techniques that are required, although, there is often a great deal of overlaps between the listed areas. In addition, the numerical solution of many mathematical problems involves some combination of some of these areas, possibly all of them. A rough categorization of the principal areas of numerical analysis is given as follows:

Systems of linear and nonlinear equations
- **Numerical solution of systems of linear equations**: This refers to solving for $x$ in the system of equation $Ax=b$ with given matrix $A$ and column vector $b$. The most important case has $A$ a square matrix. There are both direct methods of solution (requiring only a finite number of arithmetic operations) and iterative methods (giving increased accuracy with each new iteration).
- **Numerical solution of nonlinear equations**: This refers to root finding problems which are usually written as $f(x)=0$ with $x$ a vector with $n$ components and $f(x)$ a vector with $m$ components.
- **Optimization**: This refers to minimizing or maximizing a real-valued function $f(x)$. The permitted values for
\[ x = (x_1, x_2, \ldots, x_n) \] can be either constrained or unconstrained. The 'linear programming problem' is a well-known and important case; \( f(x) \) is linear, and there are linear equality and/or inequality constraints on \( x \).

**Approximation theory**

This is the use of computable functions \( p(x) \) to approximate the values of functions \( f(x) \) that are not easily computable or use approximations to simplify dealing with such functions. The most popular types of computable functions \( p(x) \) are polynomials, rational functions, and piecewise versions of them, for example spline functions. Trigonometric polynomials are also a very useful choice.

- A given function \( f(x) \) is approximated within a given finite-dimensional family of computable functions. The quality of the approximation often depends on the technique adopted.
- *Numerical integration and differentiation*: Furthermore, for approximation theory, most integrals cannot be evaluated directly in terms of elementary functions, and instead they must be approximated numerically. Although, most functions can be differentiated analytically, but there is still a need for numerical differentiation, both to approximate the derivative of numerical data and to obtain approximations for discretizing differential equations.

**Numerical Solution of Differential and Integral Equations**

These equations occur widely as mathematical models for the physical world, and their numerical solution is important throughout the sciences and engineering.

- *Ordinary differential equations*: This refers to systems of differential equations in which the unknown solutions are functions of only a single variable. The most important cases are initial value problems and boundary value problems, and these are the subjects of a number of textbooks. Of more recent interest are 'differential-algebraic equations', which are mixed systems of algebraic equations and ordinary differential equations.
This is a part of my research problems in the last decade. Also we have delay differential equations', which also is receiving attention from the Numerical Analysts.

- Solution of *Partial differential equations*: These equations occur in almost all areas of engineering, and many basic models of the physical sciences are given as partial differential equations. Thus such equations are a very important topic for numerical analysis. For example, the Navier-Stokes equations are the main theoretical model for the motion of fluids, and the very large area of 'computational fluid mechanics' is concerned with solving numerically these and other equations of fluid dynamics.

- Furthermore, *Integral equations* involve the integration of an unknown function, and linear equations probably occur most frequently. Some mathematical models lead directly to integral equations of which their solutions are difficult to obtain in closed form, hence numerical analysis is the best option to generate such solutions.

**The Concerns in Numerical Analysis**

Most numerical analysts specialize in small sub-areas of the areas listed above, but they share some common concerns and perspectives. These include the following.

- Since numerical analysis is based on approximation, it makes use of every possible assumption in order to get a solution to a problem. Thus a numerical analyst believes that if you cannot solve a problem directly, then replace it with a 'nearby problem' which can be solved more easily. This is an important perspective which cuts across all types of mathematical problems. For example, to evaluate a definite integral numerically, begin by approximating its integrand using polynomial interpolation or a Taylor series, and then integrate exactly the polynomial approximation.

- All numerical calculations are carried out using finite precision arithmetic, usually in a framework of floating-point representation of numbers. What are the effects of using such finite precision computer arithmetic? How are arithmetic calculations to be carried out? Using finite precision arithmetic will affect how we compute solutions.
to all types of problems, and it forces us to think about the limits on the accuracy with which a problem can be solved numerically.

- There is a concern with 'stability', a concept referring to the sensitivity of the solution of a given problem to small changes in the data or the given parameters of the problem. There are two aspects to this. First, how sensitive is the original problem to small changes in the data of the problem? Second, the numerical method should not introduce additional sensitivity that is not present in the original mathematical problem being solved. In developing a numerical method to solve a problem, the method should be no more sensitive to changes in the data than is true of the original mathematical problem.

- There is a fundamental concern with error, its size, and its analytic form. When approximating a problem, a numerical analyst would want to understand the behaviour of the error in the computed solution. Understanding the form of the error may allow one to minimize or estimate it. The concern is to develop algorithms that will produce a very minimal error when compared with the analytical solution of a problem. In the process of doing such, the stability of the method and the step function evaluation is put into consideration. A 'forward error analysis' looks at the effect of errors made in the solution process. This is the standard way of understanding the consequences of the approximation errors that occur in setting up a numerical method of solution.

Modern Numerical Analysis
Modern numerical analysis can be credibly said to begin with the 1947 paper by John von Neumann and Herman Goldstine, "Numerical Inverting of Matrices of High Order" (Bulletin of the AMS, Nov. 1947). It is one of the first papers to study rounding error, and include discussion of what today is called scientific computing. Although numerical analysis has a longer and richer history, "modern" numerical analysis, as used here, is characterized by the synergy of the programmable electronic computer, mathematical analysis, and the opportunity and
need to solve large and complex problems in applications. The need for advances in applications, such as ballistics prediction, neutron transport, and non-steady, multidimensional fluid dynamics drives the development of the computer and depended strongly on advances in numerical analysis and mathematical modelling.

Modern numerical analysis and scientific computing develops quickly and on many fronts. Our current focus is on numerical methods for solving ordinary differential equations, methods of approximation of functions and the impact of these developments on science and technology. Current interest is the impact of mathematical software packages, accuracy of the schemes being developed and their properties.

**Numerical Analysis and Differential Equations**

In this lecture, I will not be able to discuss all the works done on numerical analysis but at this point I will like to give some basic definitions of differential equations and their types which are useful to numerical method for solving ordinary differential equations.

Any equation containing differential coefficients such as \( \frac{dy}{dx}, \frac{d^2y}{dx^2}, \left( \frac{dy}{dx} \right)^2 \), etc is called a differential equation (Okunuga, 2008a)

Where \( \frac{dy}{dx} \) is the rate of change of y with respect to x

Examples of some differential equations include:

1. \( \frac{dy}{dx} - 5y = 0 \)
2. \( 2 \frac{d^2y}{dx^2} + xy \left( \frac{dy}{dx} \right)^2 = 0 \)
3. \( \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = 1 \)
The above examples show that there are different types of differential equations, as all of these do not belong to the same class. Hence differential equations are broadly classified into two groups:

(i) Ordinary Differential Equation (ODE)
(ii) Partial Differential Equation (PDE)

A differential equation involving ordinary derivatives (or total derivatives) with respect to a single independent variable is called an **Ordinary Differential Equation**. Examples of ODEs are Nos 1 and 2 above.

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a **Partial Differential Equation** (PDE). Examples 3 and 4 above are PDEs.

A general first order differential equation with a condition specified at the initial point is called an **Initial Value Problem** (IVP). Thus an IVP is written as

\[
\frac{dy}{dx} = f(x, y) , \quad y(x_0) = y_0
\]  

(1)

Where \((x_0, y_0)\) are the initial values which permit the solution of the differential equation to be unique.

A **stiff differential equation** is an equation with a solution involving fast decaying parameters. These basic definitions guide the areas of my research work.

**Numerical methods for solving differential equations**

Often we examine simple initial value problem of the form (1). This problem serves as the standard form for all first order IVPs from which numerical scheme can be developed. The simplest numerical method ever proposed to solve this IVP is
the one-step method of Euler. Although Euler has other methods, the explicit Euler scheme given by

\[ y_{n+1} = y_n + hf_n \]

This is known as the simplest numerical method for solving IVPs. There are other one step methods including the Runge-Kutta methods. The Euler method, though simple, is often used to generate starting values for several other methods, however, this method is not so accurate for the purpose of approximation. Several other numerical methods were developed in early days of numerical analysis but they were developed for desk computation prior to computer era.

In the attempt to develop more numerical methods, it was more and more of using numbers, approximations and error analysis. This leads to the fact that human generally appreciates numbers rather than abstract terms which may not be easily comprehended and hence not appreciated. To this end we observe that numerical analysis tends to deal with numbers as a way of presenting concrete answers to mathematical problems. This idea of numbers which many easily comprehend is also part of God's method of dealing with man.

It is noted that since Mathematics has many branches, starting with numbers, then arithmetic, later to symbols and development of equations and difficult terminologies, it got to a point that we started going back to numbers as many abstract Mathematics require meaning which can only be obtained via numbers. This is what gave birth to computer era, and today all of us use computer for our daily operations.

Do you know that everything you type on a computer is not recognised the way you input them? What computer does is to translate whatever you type into it to 0s and 1s. In other words computer codes are 0 and 1 and that is what computer uses to give you your answer. The computer re-codes everything you send into it, be it sentences or figure, into 0 and 1 and then process your request. The reason for this is that those who developed computer looked for simplest base numeral to code
it. The lowest number base to operate with is base 2 numeral which permits only 0 and 1 as its digits, rather than base ten which uses 0, 1, 2, ..., 9 as its figures. If man can do this I want to prove to you that God is a better computational mathematician that can compute with any base number and I will show this briefly.

**God is the author of Mathematics**

Mr. Vice-Chancellor sir, as an Applied Mathematician and a Numerical Analyst in particular, I apply my algorithms to treat both physical and spiritual problems. Hence, I will dwell a bit on what God says about numbers. God created numbers for use and He also work by numbers. I will give some points from the Holy Bible about how numbers and Mathematics were greatly used.

**Point 1**
Numbers are seen to be so important that the Psalmist says teach us to number our days so as to apply our hearts to wisdom. So my first submission is that wisdom emanates from numbers. Therefore everyone who loves number will necessarily be a thinker and thinkers are men who invest on wisdom.

**Point 2**
It takes a great thinker to think like God thinks. I see the way God does his counting as against that of man, which makes numbers to be the language of the thinkers: "...that with the Lord one day is as a thousand years, and a thousand years as one day." *(2 Peter 3:8)*

It seems God is not bounded by time. Event of one thousand years can be accomplished in a day. If you can think like God you can be like Him.

**Point 3**
Imagine this sequence of numbers:

1000, 10000, 19000, 28000, ... This is A.P with d= 9000

$10^3, 10^4, 10^5, 10^6, ...$ This is GP with $r = 10$

$10^3, 10^4, 10^6, 10^9, ...$ This is also a sequence
Now, Deuteronomy 32:30 says "One will chase a thousand, and two shall put ten thousand to flight".

God deals with ratio, sequences and series. The ratio is $1:10^3$ and $2:10^4$. But the series is not clear since we do not know the third term of the series, it may even not be a geometric series. So it is better to call it "Godometric" series. (a series defined by God).

**Point 4**

Do you know that God loves numbers and work specifically with certain numbers? May be the thinkers will work with such numbers.

There are some numbers that are unique in the Holy Bible. I give a list of the prominent ones. They include: 1, 3, 7, 12, 40, 70, 120.

Number "1" refers to one God or unity

Number '3' is very essential; it implies Trinity (agreement)

The number '3' describes the Trinity – The Father, the Son and the Holy Spirit. The bible says these Three are One. In God's Mathematics three equals one ($3 = 1$). The Bible says man is a triune being. He is a **spirit**, he has a **soul** and lives in a **body** (I Thessalonians 5:23). The combination of these three components make up what God calls man. Once again **three** entities equal one ($3 = 1$). In a simple mathematical equation we can write

**Spirit + Soul + Body = Man**

where **spirit** is God's breath in man, **soul** is the mind and intellect, and **body** is the shape or house for the spirit and soul to dwell. This is why God said "let us make man in our own image". The Triune God created a triune being called man (Genesis 1:26; 2:7).

This also linked with calculus idea as we can easily deduce that:

$$\int body \, d(spirit) = \text{man}$$
Do you know that calculus is in the Holy book? I will not dwell on this in this lecture but it suffice to say that John 15: 1-gave an example of integration into the Vine.

God declared himself as a triune God. God also declared himself as God of Abraham, God of Isaac and God of Jacob (no more).

The 3 are the people that God made 1-1 covenant with and declared to his people “I am the God of Abraham, Isaac and Jacob”. So number “3” seems to be important to God.

Furthermore, among all the disciples of Jesus Christ, ‘3’ of them belong to what we can call the Inner Circle. Peter, James and John were the three closest to Jesus (Luke 8: 51). These three were the ones present when special miracles were done. They were the ones at transfiguration. They were the ones when Christ went to pray at Gethsemane and more... In the transfiguration, only 3 people appeared – Jesus, Elijah and Moses. A three cord is not easily broken.

The number 3 is for completeness. Hence by mathematical induction we can state a corollary that:

The necessary and sufficient condition is that a complete man must be a three in one man (that is called Okunri meta in Yoruba)

The number “7” stands for perfection. Seven describes perfection. The Bible talks of the ‘7’ spirits of God and these seven spirits rests on Jesus for completeness (see Isaiah 11: 2, Revelation 5: 6, 4: 5).

The Lamb that was slain received 7 virtues of God for mankind. Revelations 5: 12 says: “Worthy is the Lamb that was slain the
receive (i) power, and (ii) riches, and (iii) wisdom, and (iv) strength, and (v) honour, and (vi) glory, and (vii) blessing”

Have you considered number 12? The number twelve (12) is unique in the sense that God loves working with 12. Jacob by divine agenda (I do not think it is by accident), had 12 sons which became 12 tribes of Israel till today. Exodus 15:27:

And they came to Elim, where there were 12 wells of water, . . .and they encamped there by the waters.

Solomon had 12 officers over all Israel, (1Kings 4:7).

Jesus had 12 disciples. (Luke 6:13)

In one of the miracles where Jesus fed 5000 people, we were told that “…they took up 12 baskets full of the fragments, and of the fishes” after everyone had eaten.

In the Book of Revelation alone, the number 12 was mentioned ten times, which talks of 12 gates, 12 stars 12 pearls, etc.

Point 5: The Number “Forty”
The number 40 is so significant in the Bible that we need to pay close attention to it. It is applicable to individual lives and the nation. The events of children of Israel were several described with number 40

(a) Moses life time was partitioned or collocated into three parts. The first collocation point is when he was 40 years when he came to visit his brethren. 40 years later God called him to lead his people. And for 40 years he led them through the wilderness. 40 years later after he was called by God, he died. Moses life was divided into three significant and equal 40 years. The thinkers will think about number 40.

(b) God fed the children of Israel directly from heaven with manna in the wilderness for 40 years. Miracles!

(c) For 40 years the children of Israel grieved God in the wilderness (Hebrews 3:9, 17; 8:2). These numbers call for thinking that is why this lecture is titled, Numbers: The language of the thinkers.
(d) David reigned as king for 40 years. Solomon reigned as king for 40 years. Joash reigned as king for 40 years. (1 Kings 2:11; 1 Kings 11:42; 2 Kings 12:1) Moses, Elijah and Jesus fasted for 40 days. 40 days plan and 40 day agenda can lead to some giant results, even the vision 20 – 2020 talks of 20 in the vision with any of the 20s to give 40.

Pont 6
Do you know that God uses simple mathematical operations like addition, multiplication and division? There are simple operations we use every day to get to our destination. Consider words like ‘addition’, ‘multiplication’. They sound like mathematical terms. God like using those two words very well. The word add or added appears in the Bible 47 times. The word “multiply” or similar words to it appears 87 times. The following are examples of these words:

But seek ye first the kingdom of God, and his righteousness; and all these things shall be added unto you. (Mat 6:33)

Grace and peace be multiplied unto you through the knowledge of God, and of Jesus our Lord, (2Peter 1:2)

And beside this, giving all diligence, add to your faith virtue; and to virtue knowledge; (2Peter 1:5)

My son, forget not my law; but let thine heart keep my commandments: For length of days, and long life, and peace, shall they add to thee. (Proverbs 3:1-2)

I have heard thy prayer, I have seen thy tears: behold, I will add unto thy days fifteen years. (Isaiah 38:5, 2Kings 20:6)

And the angel of the LORD said unto her, I will multiply thy seed exceedingly, that it shall not be numbered for multitude. (Genesis 16:10)
Saying, Surely blessing I will bless thee, and *multiplying* I will *multiply* thee. (Hebrews 6:14)

Point 7

**God is a computational Mathematician**

In human Mathematics, there are topics such as Differentiation and Integration. Every function that is differentiated and then integrated cannot return to the same function except under certain conditions.

John 12:24-25

"Verily, verily, I say unto you, Except a corn of wheat fall into the ground and die, it abideth alone: but if it die, it bringeth forth much fruit. He that loveth his life shall lose it; and he that hateth his life in this world shall keep it unto life eternal."

Jesus looked at this word not only from agricultural point but in a mathematical way. He is saying here that a corn will remain a corn and even will not last long before it withers away if it's not planted. A corn that is planted is in the ground and dies but with all the nutrients of the soil around it is integrated to become hundreds of seeds of corn. One corn becomes hundreds of corns. One grain of corn that dies comes alive to feed a family and a nation. This is calculus at work. The variable here is the corn which is differentiated with respect to the soil and under certain limits and condition. The process of getting the corn at harvest can be represented purely by a differential equation, which is either solved analytically or numerically. The solution that God provides here is that of a computational mathematician who will like to count the seed at the harvest because we deal with numbers as a numerical analyst.

**My Contributions to the field of Numerical Analysis**

I will like to discuss some of my contributions in the field of numerical methods, which are developed as suitable schemes for solving different classes of ordinary differential equations. I will like to divide my work into 5 segments, which will summarize most of the work which I have done to date.
The Collocation-Tau Methods

Lanczos [1956] introduced the Standard Collocation method with some selected points. However, Fox and Parker introduced the use of Chebyshev polynomials in collocating the existing method which was captioned as the Lanczos-Tau method (Fox and Parker, 1968). Also, Ortiz (1969) went on to discuss the general Tau method which was later extended by Onumanyi and Ortiz (1984), to a method known as the Collocation-Tau method. The Standard Collocation method with method of selected points provides a direct extension of the Tau method to linear ODEs with non-polynomial coefficients. The Collocation -Tau method however uses the Chebyshev perturbation terms to select the collocation points. Okunuga and Onumanyi (1985, 1986) gave the generalized Tau method which permits exact fractional values in the computation with more than one $\tau$-term as perturbation on the right hand side of the linear differential equation.

The novel approach introduced by the authors developed accurate collocation methods by various types for the solution of ordinary differential equations. For a higher order scalar differential equation, numerical results show that the proposed method is more accurate than the Standard Collocation method for the same degree of polynomial approximation. We observed that when differential equation with polynomial coefficients is involved, the new method gives identical results with the Lanczos Tau method. Furthermore, this method provides a direct approach of extending it to non-linear differential equations. Thus, Okunuga and Sofoluwe (1990) modified the Tau method to accommodate non-linear differential equations and by Newton linearization processes, this new Collocation method was applied on the problem describing the meridian of the dropped shaped tank. The problem has some singularities at the initial point which makes it more difficult to solve by many existing numerical methods. However our Collocation method applied on this problem gave some high accuracy as reported in our work.
Exponential Fitted Schemes with Composite Formulas
The author over the years introduced the exponentially fitted formulas which were found to be quite suitable for solution of stiff initial problems. Stiff differential equations usually pose some difficulties for several numerical schemes, as many of them usually fail to cope with the fast decaying nature of the solution of stiff problems. These often are problems that emanate from modern Physics and astronomical problems. Also some are due to chemical kinetic reactions. The author developed orders 2 to 6 exponentially fitted methods which is a form of the Multiderivative Linear Multistep Method (MLMM) and suitable for handling the stiff problem that permits exponential fitting. (Okunuga, 1999 (a), (b)) The methods with low orders were derived to give a unified model that combines both the predictor and the corrector methods during the process of exponential fitting. The higher order of the methods introduced some other variants such as Padé approximation so as to obtain a better result and satisfying some good stability criteria (Okunuga & Sofoluwe 2008). The methods developed in all of these papers were seen to be very accurate, convergent and satisfied the A-stability and zero stability conditions.

In further research carried out, some composite integration formula were developed and tested on systems of non-linear IVPs resulting from chemical kinetic reaction problem. The results show that order 4 scheme exhibits the highest accuracy and was recommended for use to all users of numerical methods when solving stiff problems.

In Okunuga 1998, the stability conditions for which all classes of Multiderivative Linear Multistep Method (MLMM) must satisfied were discussed including the convergence of the methods. The author showed in his work that all the orders 2 to 6 methods of the MLMM are A- stable, a condition which is difficult for many numerical schemes to satisfy.

Furthermore, the schemes satisfied the stiff stability condition and they also satisfied the definitions of zero and absolute
stabilities. The article shed some light on how accuracy can be improved upon when solving stiff IVPs. The results obtained by using our schemes when compared with results of other authors were found to be more accurate (Okunuga, 2009).

Abhulimen & Okunuga (2008) developed a more difficult scheme by extending the earlier work done by Okunuga (1999(a)). The authors developed a fifth order of the MLMM with higher derivatives that excludes the Padé approximation in lower derivative scheme earlier proposed by Okunuga & Sofoluwe (2008). The price paid for including the higher derivatives in the development of the new scheme, was justified by the accuracy of the results obtained when implemented on standard problems.

A class of two-step second derivative Linear Multistep Method with some exponential fittings for order two was developed for generating solutions for Stiff IVPs (Okunuga, 2009). The resulting integration formula is applied on systems of stiff problems in Predictor-Corrector mode.

The predictor formula corresponding to the case \( k = 2 \) with one free parameter is obtained as

\[
y_{n+2} - y_n = h\{sf_{n+2} + (2 - s)f_n\}
\]

while the corrector method obtained is given by

\[
y_{n+2} - y_n = h\{rf_{n+3} + \frac{1}{2} (2 - 3r)f_{n+2} + \frac{1}{2} (2 + r)f_n\}
\]

By introducing exponential fitting, we write the two formulas as a single formula to get

\[
y_{n+2} = y_n + h\frac{R(q) + \frac{1}{2} q(r + 2)}{1 + \frac{1}{2} q(3r - 2)} = R(q).
\]

Equation (5) unites both the predictor and the corrector formula. The formula (5) together with \( s \) and \( r \) are used to generate solutions to stiff problems for which exponential fittings are applicable.

The method derived was implemented on some standard problems and the result obtained gave a very high accuracy
compared to some known methods. This is partly due to the choice of the free parameters with the advantage of the exponential fitting.

The peculiarity of order 2 formula derived in this paper is that it is simple compared to higher order of the MLMM for example the fourth order given by Okunuga (1999a). Furthermore its accuracy is high enough compared to the exact solution.

To illustrate some of our results, the Pade Exponential scheme was used to solve some Stiff problem discussed by Dalquist.

**Problem 1**

Dalquist and Bjock (1974) showed that there are some stiff problems for which Runge-Kutta (R-K) method is unsuitable. One of such problems is the second order stiff differential equation

\[ y'' + 1001y' + 1000y = 0 \]

\[ y(0) = 1, \quad y'(0) = -1 \]

This problem has a general solution given by

\[ Y(x) = Ae^{-x} + Be^{-1000x} \]

And for solution in \([0,1]\), the exact solution is given as \( y(x) = e^{-x} \)

By using the explicit fourth order R-K method, Dalquist showed that the method explodes for a step length \( h > 0.0027 \). A similar argument is adduced by Shampine and Gladwell (2003) despite that this is unsatisfactory step size for describing the function \( e^{-x} \)

On the contrary, this same problem was solved using our fifth order method with step- lengths greater than 0.0027. The results obtained at \( x = 1 \) using both the R-K method and the Padé-Exponential Formula (PEF) are given in Table 1 below.
It was observed that the explicit R-K method could not cope with this problem for \( h > 0.0027 \). However, the order five Padé-exponential formula gave a far better accuracy than the explicit R-K scheme.

**Problem 2**

Also, we considered the Chemical Kinetic Problem (7). Gear (1971) discussed the application of stiffly-stable integration formulae based on backward difference approximation of the derivative and used the following example to illustrate the method

\[
\begin{align*}
y_1' &= -0.013 y_1 + 1000 y_1 y_3 \quad y_1(0) = 1 \\
y_2' &= 2500 y_2 y_3 \quad y_2(0) = 1 \\
y_3' &= 0.013 y_1 - 1000 y_1 y_3 - 2500 y_2 y_3 \quad y_3(0) = 0
\end{align*}
\] (7)

This problem is an application of chemical reaction kinetic in which \( y_1 \) represents the concentration of a very reactive species which is an intermediate in the course of the reaction and always stays small. \( y_1 \) and \( y_2 \) are monotonically decreasing and increasing respectively while \( y_3 \) increases to a maximum and thereafter is monotonically decreasing. This problem is considered using our fifth order Padé-exponential formula and the result obtained shows that the accuracy is high compared to the exact solution (see Table 2 below).
Table 2: Experimental results on non-linear stiff problem using the 5th order PEF

<table>
<thead>
<tr>
<th>Steplength</th>
<th>$y_1(1)$ Absolute Error</th>
<th>$y_2(1)$ Absolute Error</th>
<th>$y_3(1)$ Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0625</td>
<td>$4.4 \times 10^{-14}$</td>
<td>$4.4 \times 10^{-14}$</td>
<td>$1.3 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.05</td>
<td>$4.4 \times 10^{-14}$</td>
<td>$-4.4 \times 10^{-14}$</td>
<td>$-8.9 \times 10^{-15}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$6.7 \times 10^{-13}$</td>
<td>$6.7 \times 10^{-13}$</td>
<td>$1.8 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

It will be observed from Table 2 that for any of the step-lengths used, the method is quite accurate. Hence the integration formula is very efficient.

**Developments of Multiderivative Explicit Runge-Kutta (MERK) Methods**

The author together with his research group, also did some work on the popular Runge-Kutta methods in a bid to improve on the existing methods. It was noted that explicit Runge-Kutta (ERK) methods have some limitations among which are the maximum attainable orders of the methods. Traditionally, the maximum attainable order of an $s$–stage Explicit Runge-Kutta method is not greater than $s$. However, we developed a 2–stage ERK Method of Order 4, which is twice as high as the expected order for a 2-stage method. The paper also discussed the stability of the method (Akanbi et al., 2008 (a), (b)). We deduced that

- Four of our methods with stability functions $P_1$, $P_3$, $P_4$, $P_5$ have the same region of absolute stability ($-2.78529 < \bar{h} < 0$) with the 4 stage classical methods of order 4;
- Methods with stability functions $P_8$ ($-3.1195 < \bar{h} < 0$), $P_9$ ($-2.93842 < \bar{h} < 0$), $P_{11}$ and $P_{12}$ ($-2.78529 < \bar{h} < 0$) have wider regions of absolute stability than the 4 stage classical methods of order 4;
- Whereas 4 – stage classical methods of order 4 require four internal function evaluations, these newly derived methods require two internal function evaluations to advance from $x_n$ to $x_{n+1}$ because they are 2-stage methods, thus preferred.
The newly derived schemes given were implemented on a system of Differential – Algebraic Equation (DAE) having some singularities. Butcher (2003) discussed the difficulties that are inherent in DAEs and their solutions. It is a known fact that rigorous researches suitable for these class of problems and probably simpler methods of handling the DAEs are still on. Thus, we considered the DAE (9) given by Lambert (1991).

\[
\begin{align*}
  y' + yz &= 0 \\
  y^2 + yz + 1 &= 0 \\
  y(0) &= 0
\end{align*}
\]

(9)

The analytical solution of the DAE (1) is

\[ y(x) = \tan(x + \frac{\pi}{4}) \] , \[ z(x) = -y - \frac{1}{y} \]

This problem requires a scheme of a very high order of accuracy due to the singularity occurring at \( x = \frac{\pi}{4} \). The computational results using the methods derived in this paper were compared with the existing methods.

As far as our results showed, the newly developed algorithm was seen to be the most efficient when compared with other known ones. This is illustrated in Figures 4 below.

![Figure 4: Graph of Numerical Experiment for h = 0.1](image)
The research group went further to examine some of the properties of RK methods which were developed. One of the characteristics of selecting a good numerical algorithm for Initial Value Problems is the error bounds on the method. We examined the error bounds of the 2-stage Multiderivative Explicit Runge–Kutta methods. The choice of step lengths for this new class of methods is also discussed. Interestingly this new class of methods has higher bounds when compared with the standard explicit Runge–Kutta methods (See Okunuga & Akanbi 2008; Akanbi et al. 2008(a)).

In our work, we discussed these errors in relation to the new 2-stage Multiderivative Explicit Runge–Kutta (MERK) methods. Consequently, we tested our claims on two problems. The first one is the IVP (Fatunla, 1988; Lambert, 1973).

\[ y'(x) = -10(y(x) - 1)^2, \quad y(0) = 2, \quad 0 < x < 1 \quad (10) \]

with an allowable error tolerance \( \varepsilon = 10^{-4} \). The theoretical solution of this problem is \( y(x) = 1 + \frac{1}{10x + 1} \).

The result of the error bounds and the bounds on the step length \( h \) using our methods is compared as shown in figure 5.

![Figure 5: Error Bounds on the step length h for the IVP (10)](image)
Also, we considered the IVP [Fatunla, 1988; Lambert, 1973]

\[ y'(x) = 1 + (y(x))^2, \quad y(0) = 1, \quad 0 \leq x \leq \frac{\pi}{4} \]  

(11)

The theoretical solution of this problem is \( y(x) = \tan\left(x + \frac{\pi}{4}\right) \)

The following bounds can readily be established which are shown in figure 6.

![Error Bounds on the steplength h for the IVP (11)](image)

**Figure 6: Error Bounds on the steplength h for the IVP (11)**

The MERK methods produced a higher bound on the step length for the two numerical experiments considered. The 4 stage ERK methods of order 4 produced the highest step length bound among the classical methods implemented with the IVPs. However, the least step length bound produced by MERK4 amongst the new MERK methods is higher than the 4 stage ERK method. The implication of this is that, MERK methods will cope with certain higher values of step length \( h \) than the classical ERK methods.

Furthermore, we examined the convergence and stability properties of these new methods, which are called 2-stage
Multiderivative Explicit Runge-Kutta (MERK) methods. The methods are found to be convergent to order 3 and order 4; and they possess wider region of absolute stability than the well-known explicit Runge-Kutta methods with the same internal stages evaluation.

Similarly, a new family of 3-Stage Multiderivative Explicit Runge-Kutta Methods was developed for the solution of Initial Value Problems (IVPs) in Ordinary Differential Equations. In this work, we present the bounds on the stepsize required for the implementation of this family of methods. This bound is one of the parameter required in the design of program codes for solving IVPs. A comparison of the stepsize bound was made vis-a-vis the existing Explicit Runge-Kutta Methods using some standard problems. The computation shows that the family of methods competes well with the popular methods.

Conclusively, from the computational results, the new MERK methods possess higher stepsize bounds and they can be used to compute the solution of Initial Value Problems in Ordinary Differential Equations with higher values of stepsize $h$. (See Akanbi & Okunuga 2006; Akanbi et al. 2010).

Furthermore, we considered some problems of interest in which some applied problems were solved by our methods. Some of the problems are the Kap’s equation, Van-der-Pol equation, electrical circuit problem and the meridian of the dropped shaped tank. (See Okunuga et al., 2001, Akanbi et al., 2011).

First we considered the treatment of Kap’s equation using a new 4th order explicit Runge–Kutta method. The authors developed new family of methods called Multiderivative Explicit Runge-Kutta (MERK) Methods. This paper presents the application of a 2-stage MERK method to stiff IVP–Kap’s equation. Numerical computations show that the new method is accurate, efficient and it competes well with some standard methods.
A fourth order member of this family that is of interest in obtaining the numerical solution of Kap's equation is,

\[ y_{n+1} = y_n + \frac{1}{4} K_1 + \frac{3}{4} K_2 \]

\[
\begin{align*}
K_1 &= hf(y_n) \\
K_2 &= hf (y_n + \frac{2}{3} K_1 + \frac{2}{3} h f_y K_1 + \frac{1}{9} h^2 (f_y^2 + f f_{yy}) K_1)
\end{align*}
\]

and it is simply referred to as MERK method.

Kap's equation is a stiff system described by the equations: (Lambert, 1991)

\[
\begin{align*}
y'(x) &= -1002y(x) + 1000z^2(x), \\
z'(x) &= y(x) - z(x)\left(1 + z(x)\right), \\
y(0) &= 1, \quad z(0) = 1.
\end{align*}
\]

The exact solution is given by:

\[y(x) = \exp(-2x), \quad z(x) = \exp(-x).\]

A Program code in Microsoft Visual C++ was written to solve this stiff IVP for \(x \in [0, 10]\), using the new MERK methods and four other conventional ERK methods. The results are displayed in figures 7-10.

Two other conventional methods, the Heun's methods and the Classical R-K methods, (Fatunla, 1988) implemented and compared with the new MERK methods. The results of the computation were found to be highly accurate and consistent with minimal errors in the solution of Kap's equation. The comparison between the numerical values generated by these methods with the theoretical solution show that these new scheme compare favourably well and appear to possess higher order of accuracy than the existing ERK methods of orders 3 and 4. The comparison of the results is displayed in figures 7-10.
1.20E-08
1.00E-08
8.00E-09
6.00E-09
4.00E-09
2.00E-09
0.00E+00

1 2 3 4 5 6 7 8 9 10

RK3s3p 1.1 1.5 2.0 2.8 3.8 5.1 7.0 9.5 1.2 1.7
RK4s4p 4.5 6.1 8.3 1.1 1.5 2.0 2.7 3.7 5.1 6.9
MERK 5.6 9.2 1.4 2.2 3.4 5.1 7.6 1.1 1.6 2.4

Figure 7: Absolute Error of \( y(x) \) in the Kap's Equation for \( x = 0 \) (0.001) 10

1.40E-07
1.20E-07
1.00E-07
8.00E-08
6.00E-08
4.00E-08
2.00E-08
0.00E+00

RK3s3p 1.20E-07
RK4s4p 4.80E-08
MERK 5.14E-10

Figure 8: Root Mean Square Error of \( y(x) \) in the Kap's Equation for \( x = 0 \) (0.001) 10
We applied the Runge-Kutta method to a simple electrical network or circuit which resulted in a differential equation. The performance of the various R-K schemes was examined. Okunuga et al., (2001) discussed the implementation of some Runge-Kutta schemes to some electrical problems. The efficiency of the various R-K schemes is demonstrated using this class of problems. The accuracy obtained in some cases was satisfactory. The implementation of semi-implicit schemes on this class of problem shows no better accuracy than the other schemes.

It was observed that implicit schemes are more tedious to implement than explicit schemes and we did show that in some instances, explicit schemes have either the same accuracy as
the implicit schemes or sometimes better accuracy are earned from explicit schemes.

In a bid to improve on the deficiencies of ERK methods especially in solving second order ODE, our group went further to develop new schemes for a direct integration of second-order ordinary differential equations of the form, $y''(x) = f(x, y)$, $y(a) = y_0$, $y'(a) = y_0'$ using the Explicit Runge-Kutta-Nystrom method with higher derivatives (Okunuga et al. 2011(a)). Various numerical schemes are derived and tested on standard problems. The Higher-Order Explicit Runge-Kutta-Nystrom (HERKN) Method given in this paper is compared with the usual Explicit Runge-Kutta (ERK) schemes. Due to limitation of ERK schemes in handling stiff problems, the extension to higher order derivative is considered here. The results obtained show an improvement to ERK schemes.

We tried to improve the Runge-Kutta-Nystrom (RKN) methods by using the work of Goeken and Johnson (1999) and Akanbi et al. (2005) in which they used the method of higher derivatives as a multistep in stage evaluations to increase the order of a Runge-Kutta (R-K) method. The order condition obtained in this paper is up to order five (5) as shown in Table 3, which ordinarily should not exceed 4 for 2-stage method. This is an improvement to the work done by earlier authors.

For the derivation of a 2 Stage method, the following algebraic equations were obtained as order conditions for the HERKN methods.
Table 3.

<table>
<thead>
<tr>
<th>Order Conditions for y</th>
<th>$O(h^n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$b_1 + b_2 = \frac{1}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$b_2c_2 = \frac{1}{6}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} b_2c_2^2 = \frac{1}{24}$</td>
</tr>
<tr>
<td></td>
<td>$b_2a_{21} = \frac{1}{24}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2} b_2c_2^3 = \frac{1}{40}$</td>
</tr>
<tr>
<td></td>
<td>$b_2c_2a_{21} = \frac{1}{40}$</td>
</tr>
<tr>
<td></td>
<td>$b_2d_{21} = \frac{1}{120}$</td>
</tr>
</tbody>
</table>

Table 4.

<table>
<thead>
<tr>
<th>Order Conditions for $y'$</th>
<th>$O(h^n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$b'_1 + b'_2 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$b'_2c_2 = \frac{1}{2}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} b'_2c_2^2 = \frac{1}{6}$</td>
</tr>
<tr>
<td></td>
<td>$b'<em>2a</em>{21} = \frac{1}{6}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{6} b'_2c_2^3 = \frac{1}{24}$</td>
</tr>
<tr>
<td></td>
<td>$b'<em>2c_2a</em>{21} = \frac{1}{8}$</td>
</tr>
<tr>
<td></td>
<td>$b'<em>2d</em>{21} = \frac{1}{24}$</td>
</tr>
</tbody>
</table>

These equations were solved by maple software and eight numerical methods were generated from the two tables above and the methods are labelled as HERKN1 – HERKN8. These methods were implemented on some IVPs.

Consider the test stiff problem,

$$y'' = A y, \quad y(0) = y'(0) = 1,$$

(14)

The exact solution for $A = -1$ is given by $y(x) = \cos x + \sin x$

Numerical Solution to problem (14) using step length $h = 0.1$ and $h = 0.05$ is analyzed using Max Norm. i.e. $\text{Max} \| y_n - y(x_n) \|$.

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It is known that ERK do not perform efficiently for stiff problems. In fact many ERK schemes fail to obtain convergent solution to stiff problem; however all the HERKN methods gave convergent solution to the problem with max error given in the Table 5 above.

This method has shown that the usual practise of reduction of second order ordinary differential equation to a systems of 2 first order ordinary differential equations can be ignored and solved directly. Also, for a reduced stage evaluation we have a method with a higher order of convergence as seen from the order conditions obtained.

**Works On Linear Multistep Methods**

The idea of collocation methods of the 1980s was revisited by Okunuga & Ehigie (2009) with intention of borrowing ideas into multistep methods. We then introduced some collocation approach in the derivation of Continuous Linear Multistep Methods. This became a new trend of developing many multistep schemes. Several of my research work lately is along...
this line. The research work provided the use of both collocation and interpolation techniques to obtain the schemes. Rather than using Chebyshev polynomials as basis function as it was always done in the past, we introduced the use of direct form of power series as an alternative to the derivation of these schemes. Multistep Methods have over the years been one of the most popular and acceptable methods for generating solutions to initial value problems of Ordinary Differential Equations.

The general k-step method or LMM of step number k given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \quad \alpha_k \neq 0.$$  \hspace{1cm} (15)

where $\alpha_j$ and $\beta_j$ are uniquely determined and $h$ is called the step length.

The LMMs generate discrete multistep schemes (15) which are used for solving an IVP. However, the Continuous Collocation approach, which require collocating at some points $x_k$ of the Linear k-step method (15) is rewritten in continuous form as,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j} \hspace{1cm} (16)$$

Where $\beta_j(x)$ is now defined as a function of x and it is continuously differentiable at least once. All through our research work on multistep schemes, we developed various continuous multistep collocation methods with some collocation points taken at the grid points using some polynomials and special functions as basis function. We further constructed some other continuous schemes at some off grid points leading to hybrid methods.

The implicit continuous Hybrid Scheme at an off grid point $s = -\frac{1}{4}$ is given by
\[ Y(x) = Y_k + h \left[ \frac{(x-x_k)}{h} - \frac{7(x-x_k)^2}{6h^2} + \frac{4(x-x_k)^3}{9h^3} \right] f_k \]
\[ + h \left[ -\frac{3(x-x_k)^2}{2h^2} + \frac{4(x-x_k)^3}{3h^3} \right] f_{k+1} \]
\[ + h \left[ \frac{8(x-x_k)^2}{3h^2} - \frac{16(x-x_k)^3}{9h^3} \right] f_{k+\frac{1}{2}} \]

Evaluating at \( x_{k+1} \), we obtain a discrete implicit hybrid formula:

\[ Y_{k+1} = Y_k + h \left[ 5f_k - 3f_{k+1} + 16f_{k+\frac{3}{4}} \right] \]

Similarly for \( s = -\frac{1}{2} \), the implicit continuous Hybrid Scheme is obtained as:

\[ Y(x) = Y_k + h \left[ \frac{(x-x_k)}{h} - \frac{3(x-x_k)^2}{2h^2} + \frac{2(x-x_k)^3}{3h^3} \right] f_k \]
\[ + h \left[ -\frac{(x-x_k)^2}{2h^2} + \frac{2(x-x_k)^3}{3h^3} \right] f_{k+1} \]
\[ + h \left[ \frac{2(x-x_k)^2}{h^2} - \frac{4(x-x_k)^3}{3h^3} \right] f_{k+\frac{1}{2}} \]

and evaluating at \( x_{k+1} \), we obtain a discrete implicit hybrid formula:

\[ Y_{k+1} = Y_k + h \left[ f_k + f_{k+1} + 4f_{k+\frac{1}{2}} \right] \]

These methods were tested and found to be very accurate. Our research group went further to develop 2-step methods of hybrid type, for first order ODE and also developed some other numerical schemes that are capable for solving second order differential equation directly (Ehigie et al., 2010).
Also, a 2-step implicit continuous hybrid scheme for $s = \frac{3}{4}$ which was derived leading to a discrete implicit hybrid formula:

$$Y_{k+2} - \frac{1}{77} (64Y_{k+1} + 13Y_k) = h \left[ -\frac{1}{77} f_{k+2} + \frac{116}{231} f_{k+1} + \frac{25}{539} f_k + \frac{1024}{1617} f_{k+\frac{7}{4}} \right]$$

Also for hybrid point $s = \frac{1}{2}$, we obtain another Implicit hybrid Scheme,

$$Y_{k+2} - \frac{1}{31} (32Y_{k+1} - Y_k) = \frac{h}{93} \left[ 15 f_{k+2} + 12 f_{k+1} - f_k + 64 f_{k+\frac{3}{2}} \right]$$

Several other schemes were derived through this technique; however the accuracy, stability and convergence of the methods determine their usefulness.

The derived schemes were tested on the IVP,

$$y' = \frac{x}{y}, \quad y(0) = -2 \quad (17)$$

This problem is known to have an analytical solution,

$$y = -\sqrt{x^2 + 4}$$

The approximate solution to the IVP is solved using our newly developed Continuous Linear Multistep Schemes. We employed the use of the continuous schemes rather than the discrete ones because solutions are obtained almost directly without much difficulty.

Let the 2-Steps Continuous Implicit Hybrid Schemes (2SCIHS) derived be denoted as follows

2SCIHS1 for $s = \frac{3}{4}$, 2SCIHS2 for $s = \frac{1}{2}$, 2SCIHS3 for $s = \frac{1}{4}$ and 2SCIHS4 for $s = -\frac{1}{2}$

Also denote the 2-Step Adams Moulton Method: 2SAMM

The Schemes are compared with the analytical solution to obtain error (E) as displayed in the table 6 below:
Table 6: Table of Errors

<table>
<thead>
<tr>
<th>X</th>
<th>E2SCIHS1</th>
<th>E2SCIHS2</th>
<th>E2SCIHS3</th>
<th>E2SCIHS4</th>
<th>E2SAMM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00E+00</td>
<td>0.00E+00</td>
<td>0.00E+00</td>
<td>0.00E+00</td>
<td>0.00E+00</td>
</tr>
<tr>
<td>0.1</td>
<td>1.56E-06</td>
<td>1.56E-06</td>
<td>1.56E-06</td>
<td>1.56E-06</td>
<td>1.56E-06</td>
</tr>
<tr>
<td>0.2</td>
<td>1.28E-06</td>
<td>1.60E-06</td>
<td>2.11E-06</td>
<td>4.98E-05</td>
<td>3.07E-06</td>
</tr>
<tr>
<td>0.3</td>
<td>9.18E-06</td>
<td>1.49E-05</td>
<td>2.41E-05</td>
<td>8.85E-04</td>
<td>2.01E-02</td>
</tr>
<tr>
<td>0.4</td>
<td>1.32E-04</td>
<td>1.89E-04</td>
<td>2.81E-04</td>
<td>8.88E-03</td>
<td>9.16E-05</td>
</tr>
<tr>
<td>0.5</td>
<td>6.77E-04</td>
<td>9.29E-04</td>
<td>1.34E-03</td>
<td>5.04E-02</td>
<td>3.24E-04</td>
</tr>
<tr>
<td>0.6</td>
<td>2.28E-03</td>
<td>3.06E-03</td>
<td>4.33E-03</td>
<td>1.23E-01</td>
<td>8.20E-04</td>
</tr>
<tr>
<td>0.7</td>
<td>6.00E-03</td>
<td>7.94E-03</td>
<td>1.11E-02</td>
<td>3.06E-01</td>
<td>1.71E-03</td>
</tr>
<tr>
<td>0.8</td>
<td>1.35E-02</td>
<td>1.76E-02</td>
<td>2.45E-02</td>
<td>6.61E-01</td>
<td>3.16E-03</td>
</tr>
<tr>
<td>0.9</td>
<td>2.69E-02</td>
<td>3.51E-02</td>
<td>4.84E-02</td>
<td>1.29E+00</td>
<td>5.31E-03</td>
</tr>
<tr>
<td>1</td>
<td>4.94E-02</td>
<td>6.41E-02</td>
<td>8.80E-02</td>
<td>2.31E+00</td>
<td>8.36E-03</td>
</tr>
</tbody>
</table>

The graph of Solution and the Bar-Chart of Errors is presented below:

![Graph and Bar-Chart](image-url)
From the results above and the graphs, it is obvious that all our schemes provide good approximate solution to the problem with minimum error except the case $s = -\frac{1}{2}$ that is 2SCIH4 and they are comparable to the 2-step Adams-Moulton scheme. By the technique of this paper, we could extend the derivation to higher step methods. The schemes generated are stable and consistent.

**Contributions on Block Methods**

Okunuga et al. (2011(b), 2012) used multistep collocation techniques to develop some 3-point explicit and implicit block methods, which are suitable for generating solutions of the general second-order ordinary differential equations of the form, $y'(x) = f(x,y,y')$, $y(a) = y_0$, $y'(a) = y'_0$. The derivation of both explicit and implicit block schemes is given for the purpose of comparison of results. The Stability and Convergence of the individual methods of the block schemes were investigated and the methods were found to be zero stable with good region of absolute stability. The 3-point block schemes derived are tested on standard mechanical problems and it is shown that the implicit block methods are not so superior to the explicit ones in terms of accuracy. The 3-point explicit scheme developed is given as block methods as
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_{n+1} \\
y_{n+2} \\
y_{n+3}
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 2 \\
0 & -2 & 3 \\
0 & -3 & 4
\end{pmatrix}
\begin{pmatrix}
y_{n-2} \\
y_{n-1} \\
y_{n}
\end{pmatrix}
+ h^2
\begin{pmatrix}
1 & 0 \\
12 & 5 \\
4 & 11
\end{pmatrix}
\begin{pmatrix}
f_{n-2} \\
f_{n-1} \\
f_{n}
\end{pmatrix}
\]

while the implicit block method is given by
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_{n+1} \\
y_{n+2} \\
y_{n+3}
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 2 \\
0 & -2 & 3 \\
0 & -3 & 4
\end{pmatrix}
\begin{pmatrix}
y_{n-2} \\
y_{n-1} \\
y_{n}
\end{pmatrix}
+ h^2
\begin{pmatrix}
1 & 0 \\
12 & 5 \\
4 & 11
\end{pmatrix}
\begin{pmatrix}
f_{n-2} \\
f_{n-1} \\
f_{n}
\end{pmatrix}
\]

which is a convergent block scheme of order 6 with error constant \[
\begin{pmatrix}
-\frac{439}{4320} \\
-\frac{3479}{2880} \\
\frac{1393}{180}
\end{pmatrix}^T
\]

The Region of Absolute Stability (RAS) of the block methods in this paper are drawn and are given in Figures 13 and 14.
A code was developed for the implementation of the schemes derived in sections 3 and 4 above. The code was designed so that it determines the initial points of the starting block methods. Thereafter, it generated the values for \( y_{n+1} \), \( y_{n+2} \), and \( y_{n+3} \) simultaneously by using the block schemes directly for the explicit schemes and predictor-corrector technique for the implicit schemes using a fixed step size \( h \). With a desired accuracy, the correction to convergence allowed some choices of tolerance (TOL) limit.

**Experimental Problems**
The Van der Pol equation which describes the Van der Pol oscillator is a second order ODE,

\[
y'' = \mu (1 - y^2) y' - \lambda y, \quad y(0) = A, \quad y'(0) = B
\]

and it assumes some real positive number \( \mu \) and \( \lambda \). The problem was named after B. Van der Pol in 1926. This equation has attracted a lot of research attention both in nonlinear mechanics and control theory. This equation has no solution in terms of known tabulated transcendental function. However to generate a numerical solution for this problem, we varied the value of \( \mu \) between \( 1 \times 10^{-4} \) and \( 1 \times 10^{-20} \). The values of \( \lambda = A = B = 1 \). However, as \( \mu \) tends to zero, the IVP has the analytical solution,

\[
y(x) = \cos(x) + \sin(x)
\]

The results are presented using maximum error which is given on Table 7.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( H )</th>
<th>Explicit</th>
<th>Implicit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Max Error</td>
<td>Tol</td>
<td>Max Error</td>
</tr>
<tr>
<td>1.00E-04</td>
<td>8.78E-05</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.00E-04</td>
<td>8.44E-05</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.00E-06</td>
<td>8.44E-05</td>
</tr>
<tr>
<td>1.00E-03</td>
<td>6.52E-06</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.00E-04</td>
<td>6.50E-06</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.00E-06</td>
<td>6.50E-06</td>
</tr>
</tbody>
</table>
We further developed schemes that can be applied to a second order non-linear ordinary differential equations of Lane-Emden type (Okunuga et al., 2012). These were solved using the boundary value technique. A class of second derivative backward differentiation formula was derived from some continuous multistep schemes using the multistep collocation technique. The technique transforms the numerical methods to a system of non-linear equations represented as a tridiagonal matrix, thereby obtaining numerical solutions concurrently on the entire range of integration. General properties of the numerical method are presented as well as the stability properties. Some equations of Lane-Emden type are solved to demonstrate the efficiency of the method.

We derived some mixed boundary value methods via the multistep collocation technique. The methods obtained have been represented as boundary value methods using the representation of block schemes. Properties such as order of convergence and region of absolute stability were highlighted using tables and figures respectively. These BVMs methods were implemented on second order nonlinear ordinary differential equations of Lane-Emden's Type and their results were found to be sufficiently accurate for various values of step length.

Also we gave various classes of Block Formulas which are capable of solving different initial value problems. The paper presents different approaches which include using the Lagrangian interpolation and collocation techniques for the derivation of these methods. The class of the methods are generally given and specific methods are used for illustration in this work. The regions of absolute stability of the methods are constructed for choices of methods which may be suitable for various IVPs. The methods are tested on both stiff and non-stiff single and systems of IVPs. Some good and accurate results are obtained and the paper is quite illustrative.

In another research work we also developed a generalized 2-step continuous multistep method of hybrid type for the direct
integration of 2nd-order differential equations in a multistep collocation technique, which yields block methods (Ehigie et al. 2010). The schemes obtained were used as a single continuous form which serves as a family of formula involving \((x, s)\) such that on substitution of an off-step point \(s\) a bi-hybrid continuous scheme is obtained. The discrete equivalent is also obtained thereafter from the continuous family of formula.

The set of schemes obtained given as:

\[
Y_{k+2} - 2Y_{k+1} + Y_k = \frac{h^2}{3} \left[ f_{\frac{k+1}{2}} + 5f_{k+1} + 4f_{\frac{k+3}{2}} \right]
\]

\[
Y_{\frac{k+3}{2}} - \frac{3}{2}Y_{k+1} + \frac{1}{2}Y_k = \frac{h^2}{32} \left[ 5f_{\frac{k+1}{2}} + 3f_{k+1} + f_{\frac{k+3}{2}} \right]
\]

\[
Y_{\frac{k+1}{2}} - \frac{1}{2}Y_{k+1} - \frac{1}{2}Y_k = \frac{h^2}{96} \left[ -13f_{\frac{k+1}{2}} + 2f_{k+1} - 96f_{\frac{k+3}{2}} \right]
\]

These formulas were tested on the following second order ODEs.

1. \(y'' - y' = 0,\quad y(0) = 0,\quad y'(0) = -1,\quad h = 0.1\)

2. \(y'' = \frac{(y')^2}{2y},\quad y\left(\frac{\pi}{6}\right) = \frac{1}{4},\quad y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\quad h = 0.01\)

The results are presented below.
### Table 8: Numerical Solution and Tables of Errors for Problem 1

<table>
<thead>
<tr>
<th>Analytical</th>
<th>E_y(1/2)</th>
<th>E_y(1/4)</th>
<th>E_y(7/4)</th>
<th>E_optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000000000</td>
<td>0.00E+00</td>
<td>0.00E+00</td>
<td>0.00E+00</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.1051709</td>
<td>3.38E-04</td>
<td>3.38E-04</td>
<td>3.38E-04</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.2214028</td>
<td>7.22E-04</td>
<td>6.51E-04</td>
<td>7.95E-04</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.3498589</td>
<td>1.16E-03</td>
<td>9.25E-04</td>
<td>1.40E-03</td>
</tr>
<tr>
<td>0.4</td>
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<td>1.66E-03</td>
<td>1.15E-03</td>
<td>2.16E-03</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.6487213</td>
<td>2.23E-03</td>
<td>1.30E-03</td>
<td>3.13E-03</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.8221189</td>
<td>2.88E-03</td>
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<td>4.33E-03</td>
</tr>
<tr>
<td>0.7</td>
<td>-1.0137527</td>
<td>3.62E-03</td>
<td>1.34E-03</td>
<td>5.79E-03</td>
</tr>
<tr>
<td>0.8</td>
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<td>4.46E-03</td>
<td>1.19E-03</td>
<td>7.56E-03</td>
</tr>
<tr>
<td>0.9</td>
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<td>5.41E-03</td>
<td>8.88E-04</td>
<td>9.70E-03</td>
</tr>
<tr>
<td>1</td>
<td>-1.7182818</td>
<td>6.49E-03</td>
<td>4.05E-04</td>
<td>1.22E-02</td>
</tr>
</tbody>
</table>

### Graph of Solution and Bar Chart of Errors

![Graph of Solution and Bar Chart of Errors](image)

**Figure 14**

![Bar Chart of Errors](image)

**Figure 15**
Table 9: Numerical Solution and Tables of Errors for problem 2

<table>
<thead>
<tr>
<th>X</th>
<th>y(1/2)</th>
<th>y(1/4)</th>
<th>y(7/4)</th>
<th>analytical</th>
<th>Ey(1/2)</th>
<th>Ey(1/4)</th>
<th>Ey(7/4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.524</td>
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<td>0.0000</td>
</tr>
<tr>
<td>0.534</td>
<td>0.25871</td>
<td>0.25871</td>
<td>0.25871</td>
<td>0.25871</td>
<td>1.66E-09</td>
<td>1.66E-09</td>
<td>1.66E-09</td>
</tr>
<tr>
<td>0.544</td>
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<td>1.66E-08</td>
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<td>0.27642</td>
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<td>4.55E-08</td>
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<td>0.564</td>
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<td>1.04E-06</td>
<td>8.65E-07</td>
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<tr>
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<td>3.04E-07</td>
<td>1.46E-06</td>
<td>1.28E-06</td>
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<td>0.32218</td>
<td>0.32218</td>
<td>4.05E-07</td>
<td>1.96E-06</td>
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</tr>
<tr>
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<td>0.33156</td>
<td>0.33156</td>
<td>0.33156</td>
<td>5.22E-07</td>
<td>2.54E-06</td>
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<td>0.34101</td>
<td>0.34101</td>
<td>6.54E-07</td>
<td>3.20E-06</td>
<td>3.17E-06</td>
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</table>

The block schemes here were used as direct integration of second order differential equations without reducing it to system of first order differential equations. Numerical results show that the schemes generated converges faster on some problems than the optimal order method.

Further research work in various articles looked at how to improve on the Backward Differentiation Formula (BDF). We developed some block backward differentiation formulas based on an interpolation approach (Okunuga et al., 2008). The methods which are variable step block schemes are designed for the treatment of first order stiff initial value problems in ordinary differential equations. The stability region for the block
methods is discussed. The methods derived in this paper are tested and seen to be efficient with a better accuracy when compared with some other known methods.

There are various problems arising from different fields of studies be it in Engineering, Social Sciences or Medical Sciences. The problems often lead to differential equations which often require numerical schemes for their solutions. Some of the numerical schemes required for the solutions of such differential equations depend on the nature of the problem.

In scientific modeling, a fundamental question that arises is whether a given differential equation together with its initial conditions can be reliably used to predict the behavior of the trajectory at later times. With the assumption of the existence theorem of the solution satisfying some Lipschitz conditions (Butcher 2003) one major criterion is to look into the solution and how the solution to such Initial Value Problem (IVP) is generated.

Thus, the next three block backward differentiation schemes for \( r = 2, \frac{11}{10}, \frac{9}{10} \) respectively are given by

\[
\begin{align*}
Y_{n+1} &= \frac{1}{4} Y_{n-1} + \frac{3}{4} y_n + \frac{3}{2} hf_n \\
Y_{n+1} &= -\frac{1}{8} Y_{n-1} + \frac{9}{8} y_n + \frac{3}{4} hf_{n+1} \\
Y_{n+1} &= \frac{100}{121} Y_{n-1} + \frac{21}{121} y_n + \frac{21}{11} hf_n \\
Y_{n+1} &= -\frac{100}{341} Y_{n-1} + \frac{441}{341} y_n + \frac{21}{31} hf_{n+1} \\
Y_{n+1} &= \frac{100}{81} Y_{n-1} - \frac{19}{81} y_n + \frac{19}{9} hf_n \\
Y_{n+1} &= -\frac{100}{261} Y_{n-1} + \frac{361}{261} y_n + \frac{19}{29} hf_{n+1}
\end{align*}
\]

(18a) \hspace{1cm} (18b) \hspace{1cm} (18c)

All the schemes given in 18 (a) – (c) are Block Backward Differentiation formulae which were tested on standard problems. It was observed that the BBDF1 (\( r=1 \)) is the best in terms of performance and accuracy. The BBDF3 had the poorest results among the three methods presented.
We went further to develop some 3-point Block methods. A 3-point block method is obtained by using the Lagrangian polynomial as basis function. The Lagrangian basis function as an approximation to the solution $y(x)$ involving back values $[x_{n-1}, x_{n-2}, x_{n-3}]$ and future values $[x_{n+1}, x_{n+2}, x_{n+3}]$. Thus, the 3-point block methods obtained is given by:

**BBDF 1:**

$$y_{n+1} = \frac{-24}{35} y_{n+2} + \frac{2}{35} y_{n+3} + \frac{16}{7} y_n - \frac{6}{7} y_{n-1} + \frac{8}{35} y_{n-2} - \frac{1}{35} y_{n-3} + \frac{12}{7} h f_{n+1}$$

**BBDF 2:**

$$y_{n+2} = \frac{150}{77} y_{n+1} - \frac{10}{77} y_{n+3} - \frac{100}{77} y_n - \frac{50}{77} y_{n-1} - \frac{15}{77} y_{n-2} + \frac{2}{77} y_{n-3} + \frac{60}{77} h f_{n+2}$$

**BBDF 3:**

$$y_{n+3} = -\frac{150}{49} y_{n+1} + \frac{120}{49} y_{n+2} + \frac{400}{147} y_n - \frac{75}{49} y_{n-1} + \frac{24}{49} y_{n-2} - \frac{10}{147} y_{n-3} + \frac{20}{49} h f_{n+3}$$

These formulas were implemented on the stiff problem given by Dahlquist et al. (1974).

$$y' = 100 \sin(x - y), \quad y(0) = 0$$

is considered for step length $h = \frac{\pi}{60}$ and $h = \frac{\pi}{30}$ for $x \in [0, 2\pi]$ with analytical solution of

$$y(x) = \frac{\sin x - 0.01 \cos x + 0.01 e^{-100x}}{1.0001}.$$

The table of errors is given in Table 10 below.
Table 10: Result of the Stiff Problem

<table>
<thead>
<tr>
<th>( r = 1 )</th>
<th>( r = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h = \frac{\pi}{30} )</td>
<td>( h = \frac{\pi}{60} )</td>
</tr>
<tr>
<td>BBDF1</td>
<td>BBDF1</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>2.70e-02</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>7.03e-02</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>3.25e-02</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} )</td>
<td>1.21e-02</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>9.56e-03</td>
</tr>
<tr>
<td>( \pi )</td>
<td>1.27e-02</td>
</tr>
<tr>
<td>( \frac{7\pi}{6} )</td>
<td>5.41e-03</td>
</tr>
<tr>
<td>( \frac{4\pi}{3} )</td>
<td>3.75e-03</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>2.77e-02</td>
</tr>
<tr>
<td>( \frac{5\pi}{3} )</td>
<td>2.38e-03</td>
</tr>
<tr>
<td>( \frac{11\pi}{6} )</td>
<td>6.26e-04</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>1.08e-03</td>
</tr>
</tbody>
</table>

From Table 10 the method BBDF2 seems to be the more accurate with step length \( \frac{\pi}{60} \) but on reduction of the step length to \( \frac{\pi}{30} \) the BBDF1 gains better accuracy leaving the BBDF2 consistent as step length reduces.

The stability of the methods in terms of RAS was investigated with BBDF1 shown to be more robust. It is observed that each method has a special attribute which may determine its choice for the solution of stiff equations.

The BBDF1 is best used when the step length \( h \) is very small as this method responds to reduction of step length by gaining more accuracy, while the BBDF2 has shown itself to be more accurate for large step length \( h \) and remains consistent even
on the reduction of $l_i$. Since the BBDF3 is unstable from the RAS so obtained, it shows that it may be unable to handle any of the stiff problems.

**Recommendations**
Mr Vice-Chancellor Sir, having worked on numerical methods for solving stiff Initial value problems and Differential algebraic equations with emphasis on the importance of numbers, it is necessary for me to make the following recommendations.

- This is the age of computer and it is expedient for everyone to be computer compliant, hence every member of staff of the University be it academic or non teaching (working in an office) should be efficient in Microsoft office especially the Microsoft word and Excel spreadsheet. It enhances the efficiency and rating of the University. This may necessitate training for all the members of staff especially the non-teaching staff.

- Apart from making Mathematics compulsory as an entry qualification for students, all year one student in the university should be made to go through Foundation Mathematics and Statistics as a general course.

- The fact that many people are not numerate makes them to while away precious hours that they could have used to contribute to the growth of the University. The way to do this is to make every one account for or justify the work done per day or per week.
Acknowledgements

Mr Vice-Chancellor Sir, I want to give all glory to the King of kings and Lord of lords, who spared my life to deliver this lecture. I owe everything to my Lord Jesus who is the author and finisher of my faith and who endowed me with brain for Mathematics even to the uppermost state of my carrier. To him alone I bow and worship.

Let me start by apologising that if your name is not listed in this book or mentioned, it is not because I forgot you, it is simply because you are probably too mathematical than myself or you operate at a level higher than this paper. However in my bigger book within my heart your name is definitely there and I appreciate you all.

I want to pay special thanks to my parents Pa. E. A. Okunuga and Mama F. O. Okunuga for giving me good training in life. They sent me to school with all my brother and sisters and put me on the path of good living. You have been so loving and wonderful to me. You provided as much as you can in my school days. You trained all of us to see that we become people in the society. I thank God that I could make you proud today. Though both of you are old, thank God for giving you long and healthy life to witness this day. Also my mother in-law, Mama J. A. Ajiboye, is a great mother. I thank you for all your prayers for me. I say, I love you all.

All through my school days, I cannot forget my teachers from primary to secondary and then to the university. In particular I want to appreciate my Headmaster at St John’s School, Aroloya, Lagos, Mr Shole who instilled discipline in us, along with my teachers from infant 1 to Standard 2 then to primary 6, especially Mr J. O. Samuel, Mr D. O. Adelaja and Mrs Talabi. At The Apostolic Church Grammar School, Ketu, I cannot forget my Principal, a highly disciplined man, Revd Ven. V.S. Adenugba (always in white jacket), who taught us the way of God and hard work. Also my Mathematics teacher in Form 5, Alhaji A. Okuboyejo, who did not just teach us Mathematics but taught us four subjects: Mathematics, Additional Mathematics,
Chemistry and Statistics. He is such an intelligent and caring man. To his credit, all students during my set passed Mathematics. Thank you Sir. During my undergraduate days at University of Ilorin, I had good and seasoned lecturers who kindled my interest in Mathematics and statistics and brought me to an enviable status at my first degree graduation. I cannot forget Late Prof. Bangudu, Prof. M.A. Ibiejugba, Prof. Osanaiye, Prof. M.S. Audu, Dr Jain, Dr Mahankti, and Dr. E. A. Adeboye (now Pastor Adeboye the General Overseer of RCCG). Dr. Adeboye did not only teach me Applied Mathematics which laid the foundation of my interest in differential equations but also selected me among the 12 disciples as inner circle with him and with much of spiritual impartation that opened up my life. I give my unreserved ovation to these men.

I will also like to specially thank my supervisor at Masters level, Prof. P. Onumanyi. He did persuade me to stay in the academics. As a Mobil oil scholar, I was to take up a job in the oil industry when he encouraged me to stay back and pursue doctorate degree. He taught me how to write journal articles and my first two academic papers (local and international) were jointly written with him and under his supervision. I am his first academic and postgraduate son. Today he is the father of numerical analysis in Nigeria.

Although I mentioned our amiable late Vice-Chancellor, Prof. Sofoluwe, (Uncle Toks), I am most- pained today because he could not witness this day in my life. Everything he did for me was only good and nothing bad. He was my mentor, father, uncle, brother and friend and I followed him closely by instruction. I am the first Ph.D. student that he supervised. He personally brought me to work in UNILAG at a time I did not want to come. He had special interest in my research and from his publications, we have jointly written more articles together than he did with anyone else. I will forever remember this great man who was loved by so many. Adieu!
To my friends, I appreciate you all for believing in me and being friends all through this journey. My thanks go to my classmates, Elder Tayo Odebamike, Mr Obidairo, Mr Soji Odukoya (ICAN), including the TACGS Old Students Association. Friends from College of Education (Tech), Akoka, Mr Ajayi, Mrs Etuk-iren, Dr. Adedayo and many more.

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I will like to thank all my students whom I have had opportunity to teach over my 28 years as a lecturer in the university. Many of you who graduated in the first few sets of LASU are still good friends till today. I love you all especially, Mr. Dosunmu, Dr. Akinwande, Dr. Ogabi, Mr Akintola, Mrs Yinka Ogunlola, Mrs Macaulay, Mr Agbolade and many more.

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Before I round up this acknowledgements, I want to specially thank the Vice-Chancellor Prof. R. A. Bello who never knew me before but took keen interest in me and bestows much trust in me. Sir, I enjoy working with you. I equally thank the two DVCs: Prof. B. I. Alo and Prof. Duro Oni you are good leaders
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Despite all my interest in Mathematics, my primary commitment is to God’s work. My motto when I was in the University is “God’s work first”. To this end I found myself in the work of church ministry and beyond. I therefore want to thank my Christian fathers and fellow ministers especially Rev. M.A. Oladokun, Rev. Monday Omobhude, Rev. Azuka Ogbolumani, Pastor Aboyeyeji, Rev Abraham Akinola Rev Johnson Oyejide, Rev Adeniyi, Rev Oset, Rev Amos Alabi, Pastor Ojo, Dr Ogunmade and several other ministers that we have worked together. I appreciate the Shittus (U.K.) and my co-workers in Word Increase Ministries: Abraham Awotunde, Osita, and all the distributors of Voice of Faith Magazine. Thanks to all the brethren of Unilorin Christian Union Alumni Fellowship. God bless you all.

To my siblings and family members, I want to thank all of you. You all have shown love to me and believe in me. I enjoy you all without any regret. I want to specially thank Mr Ayeni and my late sister Mrs Folake Ayeni, my brother and his wife, Mr. and Mrs. Eddy Okunuga, Mr. and Mrs. Ayanlowo, Mrs. Taiwo Ogundare, and our last twins Sola and Kenny and their husbands. Pastor and Dr. (Mrs) Kunle Olumade, My in-laws, Mrs Adeogun, Mr. and Dcns Awotunde, Pastor and Dcns Ajiboye, Mr. and Mrs. Jide Ajiboye, Dcn. S. Ajiboye and entire Ajiboye’s family, I love you all.

Finally I want to thank my children, Ibukun, Damilola and Melody. God gave us these wonderful children. They are always supportive and show understanding. They believe in our struggles and discipline and it has been helpful to them. Finally, I will like to thank my lovely wife, Toun, who gave me all the support since the day we married. I got her as a precious gift from God. She sacrificed all her time to train the children and allowed me to be established in my carrier,
church and ministry works before she could start her own doctoral programme. I love you my dear.

Mr Vice-Chancellor Sir, This is my story, this is my Song. Thank you all for listening.
REFERENCES


Okunuga S. A. (2008b), one leg multistep method for numerical integration of periodic second order initial value problems. *Journal of Nigerian Association of Mathematical Physics* 13, 63-68,


Wikipedia, the free encyclopaedia